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Numerical Solution to Stiff Differential Equations

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ABSTRACT

This work presents the numerical solution of stiff differential equation using Euler method. Stability region shows the stability of the numerical solution, and order finger star shows the order of convergence of the numerical solution. Stability regions are plotted to show the application of the explicit method as it is needed in real time situation. The rate of theoretical and numerical order of convergence are derived. Experimental results are described to specify the performance of both numerically and graphically based methods on the metrices such as amplification factor, stability function, stability region, theoretical and numerical rate of order of convergence, absolute and relative errors, percentage of numerical solution accuracy and local and global truncation errors.
Keywords: Stiff problems, Euler's method, Stability region, Order star finger, Stiff differential equation.

1. INTRODUCTION

The mathematical model for the tajectory y(t) path with respect to time of an aircraft during landing from space to land is given in (1).

$$y'(t) = f(t, y(t)),\ t \in [a,b],\ y(a) = \phi \tag{1}$$

Subject to the condition is $-f_y(t,y(t)) \geq \alpha > 0, t \in [a,b]$. It is a problem of calculating the shape of the unknown curve which starts at a given point t = a and satisfies the given stiff differential equation. [1] explained notion and computational aspects of stiffness. It is a collection of information provided by many scientists and engineers on stiffness. Numerical methods approximate solution to complicated problems or the problems that cannot be solved by analytical means. Researches should be conducted in other areas like structural or rock mechanics, biology, chemistry or physics. A rational approach must be defined and adopted before attempting to solve these problems. In 1769-70, [2] Leonhard Euler in his book, Institutionum Calculi Integralis, designed a numerical method for the solution of the problem (1) and it is in the form (2).

$$y_{i+1} = y_i + hf(t_i, y_i), i = 0,1,2,.....,N-1,\ y_0 = \phi \tag{2}$$

where $\{a = t_0, t_1, t_2, ..., t_N = b\}$ is a sequence of points in [a, b], using the step size h, $t_i = t_0 + ih$ and $h = t_{i+1} - t_i, i = 0,1,2,...,N-1$. The method (2) is consistent with the problem (1) as step size h approaches zero ($h \to 0$). In the literature for stiff problems, method (2) follows order one, i. e.,

1. The error per step is proportional to the square of the step size.
2. The error at a given time is proportional to the step size.

To design a numerical method, a deep knowledge about the stability of the difference equation must be known to the designer and some tips for the designers are given by [3, 4]. [5, 6] examined the numerical solution technique development by identifying the problem to the never-final preparation of automatic codes for the solution of classes of similar problems. Certain problems are discussed that includes a part along the development path. Euler's forward method serves as the basis to construct more complex numerical methods. Complex numerical methods have been generated from Euler's forward method [7-24]. A program in Maple is presented to solve initial value problems numerically [25]. A program in Matlab is presented to solve second order initial value problem numerically [26]. For ordinary differential equations, an analysis is made [27]. In the literature of stiff problems,

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the theoretical rate of order of convergence is derived. The procedures adopted to find error estimate cannot be applicable to higher order methods. So it motivates to derive the theoretical rate of order of convergence and this procedure must be applicable to higher order methods. Numerical rate of order of convergence is not derived in the literature but it is derived in this paper. For the completeness of this paper, both deivations are included. Finally, stability regions, order star, order star finger region, relative stability region, relative absolute region and order graph are not explained in the literature for stiff problems whereas this study explains all these concepts. In brief, this paper presents a detailed analysis on Euler's forward method for the numerical solution of the first order stiff differential equations. In section 2, stability analysis is given and in section 3, stability functin, stability region, order star fingers for stiff problem are provided. In section 4, both stiff and non-stiff problems are analyzed using stability functin, stability region and order star fingers. In sections 5 and 6, the theoretical and the numerical rate of order of convergence is derived accordingly. Finally section 7 shows the performance of the method, both numerically and graphically.

Based on the definite integrals [27, 28], the numerical method is designed by Euler. Based on the works of [29, 30], analysis is done in this paper. The theoretical and numerical rates of order of convergence proposed coincide with the works of singular perturbation problems by taking $\varepsilon = 1$ and for non-stiff problems [30] by considering $f_y(t,y(t)) \geq \alpha > 0, t \in [a,b]$. This method is applied to boundary value problem [31, 32]. In this work, C is a constant independent of i and h.

2. STABILITY RESULT

In this section, the stability analysis of the stiff problem is discused based on the work [33]. By applying (2) in the Dahlquist problem, (3) is obtained.

$$y' = -\lambda y, t \in [0,1],\ \lambda > 0,\ y(0) = 1 \tag{3}$$

whose exact solution is $y(t) = exp(-\lambda t)$. (4) states the difference equation.

$$\frac{y_{i+1}}{y_i} = 1 - h\lambda \tag{4}$$

The corresponding amplification factor $Q(\lambda h)$ is given in (5).

$$Q(\lambda h) = 1 - h\lambda \tag{5}$$

Stability: A numerical method is stable if (6).

$$Q(\lambda h) \leq 1 \tag{6}$$

Using (6), one can determine the maximum step size h for which the method is stable. Equation (6) holds only when $0 \leq h \leq \frac{2}{|\lambda|}$.

3. STABILITY REGION

In this section, the stability region of the solution of stiff problem is presented both theoretically and figuratively based on the work [33] of stiff problems.

Stability region of exact solution: The stability function of the exact solution of the Dalquist problem (3) is given in (7).

$$R(z) = exp(-z) \tag{7}$$

where $z = h\lambda$, z is a complex number and the stability region is the entire right half of the complex plane as in figure 1.

Stability region of numerical solution: By applying Euler's method (2) to the problem (3), the stability function of the numerical solution is given as in (8).

$$R(z) = 1 - z \tag{8}$$

where $z = h\lambda$, z is a complex number and the stability region is a region inside unit circular region with center (1, 0) in the right half complex plane which is presented in figure 2.

By applying the Euler forward method to (3), the numerical solution is stable inside the contour and unstable, if the values of $z = h\lambda$ is outside the region $\{z \in C / |1 - z| \leq 1\}$.

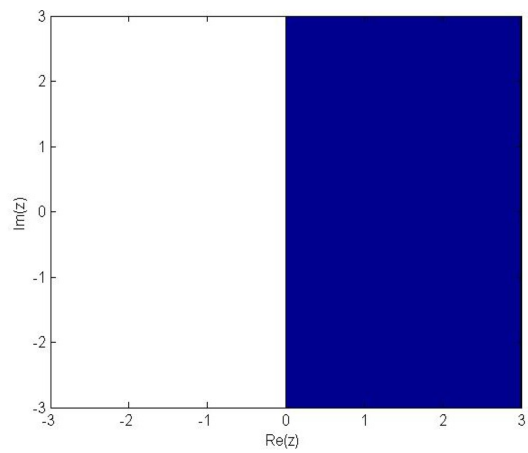

Figure 1.Stability region of exact solution of stiff problem.

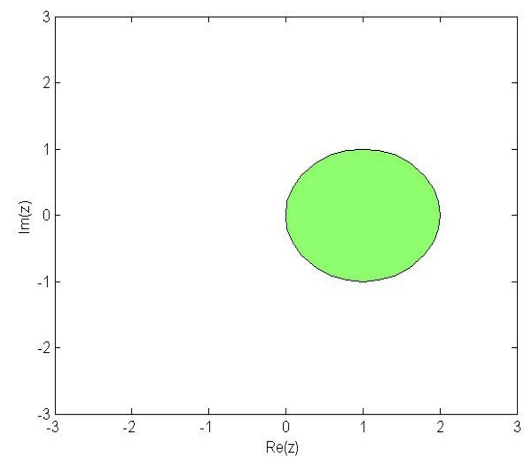

Figure 2.Stability region of numerical solution of stiff problem

Order star finger: The degree of the stability function of the stiff problem by applying Euler's method is one and so the order of the numerical method is one. In figure 3, only one finger is inside the stability region and so the method is of order one.

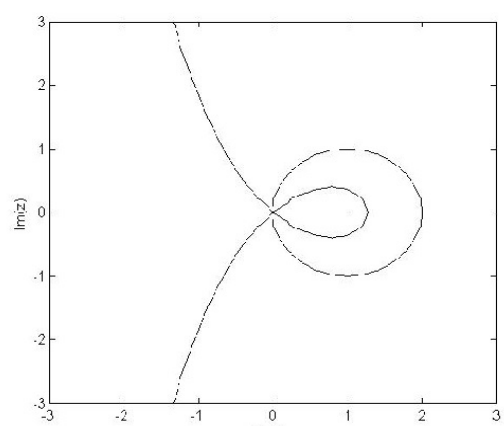

Figure 3.Order graph of Euler's forward method for stiff problem

Order star finger region of numerical method: The order star finger region of Euler's method (2) is given in figure 4.

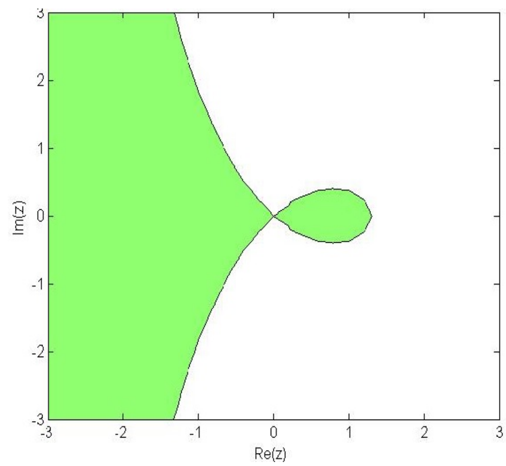

Figure 4.Order star-finger of numerical method

Relative stability region: The relative stability region or the first kind order star of the Euler's forward method is given in figure 5.

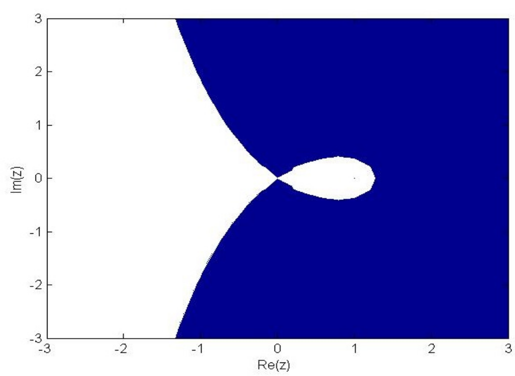

Figure 5.Relative stability region of order star finger numerical method

Absolute relative stability region of numerical method: Absolute stability region of the Euler forward method obtaining relative comparision with one is given in figure 6.

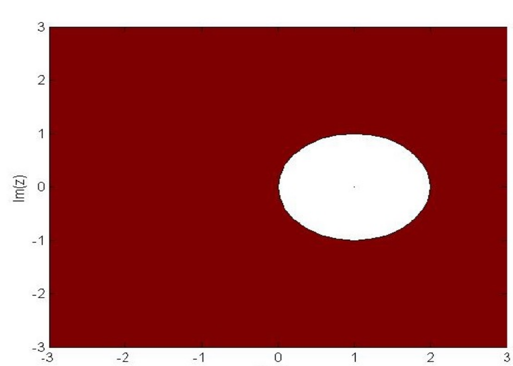

Figure 6.Absolute relative stability region of numerical method

4. STABILITY REGION STIFF AND NON-STIFF PROBLEMS

One can identify and conclude the stability region of the Euler's method for both stiff and non-stiff problems (3) from the figure 7.

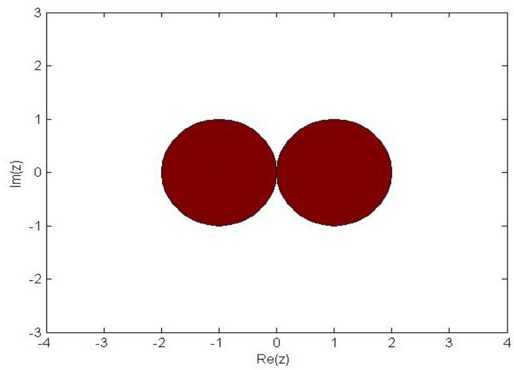

Figure 7.Stability region of Euler's forward method -stiff and non-stiff problems

From figure 8, there is only one finger inside the stability regions which shows that the Euler's forward method is of order one for both stiff and non-stiff problems.

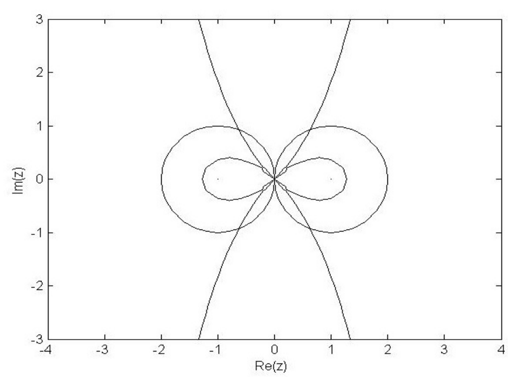

Figure 8.Order of Euler's forward method -stiff and non-stiff problems

5. THEORETICAL RATE OF ORDER OF CONVERBENCE

In this section, the theoretical rate of order of convergence of the numerical method is derived. The following theorem states the main result of this section.

If y(t) is the solution of the given differential equation (1) and y_i is the numerical solution of the Euler's method (2), then, for i = 0(1)N, an error estimate of the form given in (9) is applied.

$$|y(t_i) - y_i| \leq Ch^1 \tag{9}$$

where C is independent of i and h.

Proof: The solution y(t) of (1) at the nodal point $t = t_{i+1}$ can be expressed in terms of solution y(t) at the nodal point at $t = t_i$, using Taylor's series given in (10).

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + O(h^3), y(t_0) = \phi \tag{10}$$

where $h = t_{i+1} - t_i, i = 0, 1, ...N - 1$.

The numerical method (2) can be expressed as in (11).

$$y_{i+1} = y_i + hf(t_i, y_i), y_0 = \phi \tag{11}$$

where $h = t_{i+1} - t_i$ and $y_i = y(t_i), i = 0, 1, ...N$.

(11) can be rewritten as (12).

$$y_{i+1} = y_i + hy'(t_i), y_0 = \phi \tag{12}$$

From (10) and (12), can be defined as in (13).

$$y(t_{i+1}) - y_{i+1} = y(t_i) - y_i + \frac{h^2}{2}y''(t_i) + O(h^3)$$

$$e_{i+1} = e_i + \frac{h^2}{2}y''(t_i) + O(h^3), e_0 = 0 \tag{13}$$

where $e_i = y(t_i) - y_i, i = 0, 1, ...N$. The error per step, Local Truncation Error (LTE) is given in (14).

$$LTE = \frac{h^2}{2}y''(t_i) + O(h^3) \tag{14}$$

The error at a given time t is termed as Global Truncation Error [GTE] and is obtained from (15) by rewritting (15) as follows.

$$D_+e_i = \frac{h^1}{2}y''(t_i) + O(h^2)$$

$$GTE = \frac{h^1}{2}y''(t_i) + O(h^2) \tag{15}$$

From (14) the error estimate for the absolute error is determined. For i = 0, 1, 2,...,N-1, given in (16)-(18).

$$e_1 = e_0 + \frac{h^2}{2}y''(t_0) + O(h^3), e_0 = 0 \tag{16}$$

$$e_2 = e_1 + \frac{h^2}{2}y''(t_1) + O(h^3), \tag{17}$$

$$e_i = e_{i-1} + \frac{h^2}{2}y''(t_{i-1}) + O(h^3), \tag{18}$$

continuing as such finally,

$$e_N = e_{N-1} + \frac{h^2}{2}y''(t_{N-1}) + O(h^3) \tag{19}$$

$$e_N = \frac{h^2}{2}\sum_{j=0}^{N-1} y''(t_j) + O(h^3) \tag{20}$$

Let $k = max\left|y''(t_j)\right| for j = 0(1)N-1$, then,

$$|e_N| \leq \frac{h^2}{2}\sum_{j=0}^{N-1}\left|y''(t_j)\right| + O(h^3) \tag{21}$$

$$|e_N| \leq \frac{h^2}{2}\sum_{j=0}^{N-1} k + O(h^3)$$

$|e_N| \leq \frac{h^2}{2}Nk + O(h^3)$, since $N = \frac{b-a}{h}$,

$$|e_N| \leq Ch \tag{22}$$

Now for i = 0 and i = N, an estimate is defined. And for i = i, 2,..., N-1, the error estimate by taking (19) is found out.

$$e_i = \frac{h^2}{2}\sum_{m=0}^{i-1} y''(t_m) + O(h^3)$$

$$|e_i| \leq \frac{h^2}{2}\sum_{m=0}^{i-1} \left|y''(t_m)\right| + O(h^3)$$

since,

$$\sum_{m=0}^{i-1} \left|y''(t_m)\right| \leq \sum_{j=0}^{N-1} \left|y''(t_j)\right|$$

$$|e_i| \leq \frac{h^2}{2}\sum_{j=0}^{N-1} \left|y''(t_j)\right| + O(h^3) \tag{23}$$

The right hand side of (22) is same as in (24), and so,

$$|e_i| \leq Ch \tag{24}$$

for i = 1(1)N-1. Finally, from (17), (23) and (25), the desired estimate (9) follows for i = 0(1)N,

$$|e_i| \leq Ch \tag{25}$$

Hence the proof.

Theoretical rate of order of convergence: If y(t) is the solution of the given differential equation (1) and y_i is the numerical solution of the numerical method. And, if we have an error estimate of the form,

$$max_{0\leq i\leq N} |y(t_i) - y_i| \leq Ch^n \tag{26}$$

then n is the theoretical rate of order of convergence of the method and it can be obtained from the estimate (@7). Rewriting (27) as,

$e_i^h = Ch^n$ and $e_{2i}^{\frac{h}{2}} = C\frac{h^n}{2^n}$ then taking the ratio, the theoretical rate of order of convergence of a numerical method is obtained as,

$$n = \frac{\log(\frac{e_i^h}{e_{2i}^{\frac{h}{2}}})}{\log 2} \tag{27}$$

where $e_i^h = y(t_i) - y_i^h$ and $e_{2i}^{\frac{h}{2}} = y(t_{2i}) - y_{2i}^{\frac{h}{2}}$. Here y_i^h stands for the numerical solution obtained using step size h and $y_{2i}^{\frac{h}{2}}$ stands for the numerical solution obtained using step size $\frac{h}{2}$. From (28) and (9) the theoretical rate of order of convergence of Euler's method is of one (n = 1). When an exact solution is not known, finding out the numerical rate of order of convergence is the next task. To this, an alternate numerical rate of order of convergence is derived in the next section.

6. NUMERICAL RATE OF ORDER OF CONVERGENCE

Derivation of numerical rate of convergence order is given, where the following theorem provides the main result of this section.

Let y(t) be the solution of the given differential equation (1), and y_i^h and $y_{2i}^{\frac{h}{2}}$ be the numerical solution of the Euler's method (2) using step sizes h and $\frac{h}{2}$ respectively. Then, for i = 0(1)N, there exists an error estimate of the form,

$$\left|y(t_i) - y_i^h\right| = \left|2[y_{2i}^{\frac{h}{2}} - y_i^h]\right| \leq Ch^1 \tag{28}$$

where C is independent of i and h.

Proof: Let $w_i = 2[y_{2i}^{\frac{h}{2}} - y_i^h]$ for i = 9(1)N. For i = 0; w_0 = 0; and for i = 1(1)N, from (2),

$$w_{i+1} = w_i + h[f(t_{2i}, y_{2i}^{\frac{h}{2}}) + f(t_{2i+1}, y_{2i+1}^{\frac{h}{2}})], w_0 = 0 \tag{29}$$

Using the procedure in theorem. 5.1, (30) gets the form,

$$w_{i+1} = w_i + \frac{h^2}{2} y''(t_i) + O(h^3) \tag{30}$$

From (31), the error per step is given by,

$$LTE = \frac{h^2}{2} y''(t_i) + O(h^3) \tag{31}$$

(31) can be rewritten as,

$$D_+ w_i = \frac{h^1}{2} y''(t_i) + O(h^2)$$

and hence the error at a given time t is given by,

$$GTE = \frac{h^1}{2} y''(t_i) + O(h^2) \tag{32}$$

Now, (31) is for i = 1(1)N. Adopting the procedure followed in theorm. 5.1, for i = 1(1)N,

$$w_i = w_{i-1} + \frac{h^2}{2} y''(t_{i-1}) + O(h^3) \tag{33}$$

Now, for i = 1(1)N, a relation can be stated as in (34).

$$w_i = \frac{h^2}{2} \sum_{m=0}^{i-1} y''(t_m) + O(h^3) \tag{34}$$

Following the procedure in theorm. 5.1,

$$w_i \leq \frac{h^2}{2} \sum_{m=0}^{N-1} y''(t_m) + O(h^3) \tag{35}$$

Taking absolute value on both sides,

$$|w_i| \leq \frac{h^2}{2} \sum_{m=0}^{N-1} \left|y''(t_m)\right| + O(h^3) \tag{36}$$

Let $k = \max |y''(t_m)|$, form = 0, 1, 2,....N-1, then, for i = 1(1)N,

$$|w_i| \leq \frac{h^2}{2} \sum_{m=0}^{N-1} k + O(h^3) \leq \frac{h^2}{2} NK + O(h^3 \leq \frac{h^1}{2(b-a)} k + O(h^3) \tag{37}$$

From (30) and (38), the required result is obtained for all i = 0(1)N.

$$|w_i| \leq Ch \tag{38}$$

Hence the proof.

Numerical rate of order of convergence: Let y(t) be the solution of the given differential equation (1) and y_i^h and $y_{2i}^{\frac{h}{2}}$ are the numerical solution of the numerical method using step sizes h and $\frac{h}{2}$ respectively. Then, an error estimate of the form (39) exists.

$$max_{0\leq i\leq N}\left|y(t_i)-y_i^h\right| = max_{0\leq i\leq N}\left|\frac{2^n}{2^n-1}[y_{2i}^{\frac{h}{2}}-]\right| \leq Ch^n \tag{39}$$

Here n is the numerical rate of order of convergence of the method and it can be obtained from the estimate (40). Rewriting (40) as,

$w_i^h = Ch^n$ and $w_{2i}^{\frac{h}{2}} = C\frac{h^n}{2^n}$ then taking the ratio, the numerical rate of order of convergence of a numerical method is obtained as in (40),

$$n = \frac{\log(\frac{w_i^h}{w_{2i}^{\frac{h}{2}}})}{\log 2} \tag{40}$$

where $w_i = 2[y_{2i}^{\frac{h}{2}} - y_i^h]$

$$w_{2i} = 2[y_{4i}^{\frac{h}{4}} - y_{2i}^{\frac{h}{2}}]$$

Here $y_{2i}^{\frac{h}{2}}$ stands for the numerical solution obtained by using step size $\frac{h}{2}$ and $y_{4i}^{\frac{h}{4}}$ stands for the numerical solution obtained by using step size$\frac{h}{4}$. From (40) and (41), the numerical rate of order of convergence of Euler's method is of one (n = 1).

7. EXPEIMENTAL RESULTS

In this section, both numerical and graphical results for stiff problem (3) with λ = 1 are given in the interval [0, 1].

Theoretical rate of order of convergence: The theoretical rate of order of convergence of a numerical method P_N when the exact solution is known is defined as,

$$P_N = \frac{\log(\frac{e_i^h}{e_{2i}^{\frac{h}{2}}})}{\log 2}$$

where $e_i^h = y(t_i - y_i^h)$ and $e_{2i}^{\frac{h}{2}} = y(t_{2i} - y_{2i}^{\frac{h}{2}}$. Here, N refers to number of nodal points on using a particular step size h. It is observed from the table 1 that the theoretical rate of order of convergence is one.

Table 1.Theoretical rate of order of convergence

h	N	maximum absolute error	theoretical rate of convergence P_N
2^{-2}	4	2.427063E-02	1.039626424
2^{-3}	8	1.1806531E-02	1.019465221
2^{-4}	16	5.824152E-03	1.009526567
2^{-5}	32	2.892917E-03	1.064728925
2^{-6}	64	1.441725E-03	-
2^{-7}	128	1.441725E-03	-
2^{-8}	2565	1.441725E-03	-
2^{-9}	256	1.441725E-03	average rate $P = \frac{1}{4}\sum P_N = 1:033336783$

Numerical rate of order of oonvergence: The numerical rate of order of convergence of a numerical method P_N when the exact solution is not known is defined as,

$$P_N = \frac{\log(\frac{w_i^h}{w_{2i}^{\frac{h}{2}}})}{\log 2}$$

where $w_i = [y_{2i}^{\frac{h}{2}} - y_i^h]$ and $w_{2i} = [y_{4i}^{\frac{h}{4}} - y_{2i}^{\frac{h}{2}}]$ Here N refers to the number of nodal points on using a particular step size h. It is observed from table 2, that the numerical rate of order of convergence is one. From tables 1 and 2, it is observed that using problem (3) with λ = 1, the rate of order of convergence is one as resulted.

Table 2.Numerical rate of order of convergence

h	N	maximum absolute error	numerical rate of convergence P_N
2^{-2}	4	2.493043E-02	1.05941055
2^{-3}	8	1.196232E-02	1.028916676
2^{-4}	16	5.882470E-03	1.028990505
2^{-5}	32	2.902383E-03	1.014270724
2^{-6}	64	1.444078E-03	1.007089241
2^{-7}	128	7.202727E-04	1.003533541
2^{-8}	256	3.596963E-04	average rate $P = \frac{1}{6}\sum P_N = 1 : 023701873$

Order graph: The order graph for the solutions of the problem (3) with λ = 1 and the numerical method is given in figure 9. The curve with + sign refers to numerical solution and the curve with * sign refers to exact solution. It is a plot of h and Q(h). With respect to exact solution of problem (3), Q(h) = exp(-h), and with respect to the Euler's method Q(h = 1 - h. From figure 9, it is observed that, the curve with respect to numerical solution deviates downwards from the curve with respect to the exact solution as time step progresses.

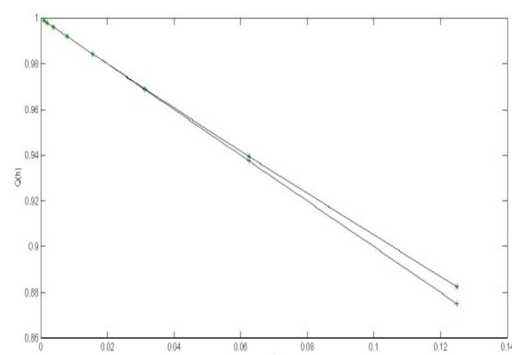

Figure 9.Order graph of Euler's forward method for stiff problem

Numerical solution of stiff problem: In figure 10, curves with + sign, . sign and * sign represent exact solution, numerical solution and the absolute error respectively. It is observed from figure 10, that as the absolute error gets deviated away upwards from the t-axis, the numerical solution gets deviated downwards away from the exact solution. If the absolute error is very closer to the t-axis as time get increased, then the numerical solution comes closer to the exact solution.

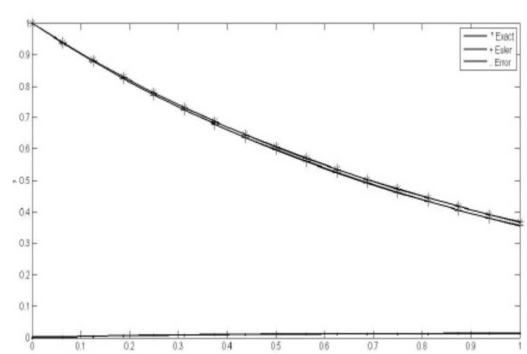

Figure 10.Exact solution, numerical solution and absolute error

In figure 11, curves with * sign and + sign represent absolute error and relaive error respectively. It is observed from figure 11, as the absolute error gets deviated away upwards from the t-axis, the relative error gets deviated upwards away from the absolute error. If the absolute error is very closer to the t-axis as time gets increased, then the relative error comes closer to the absolute error.

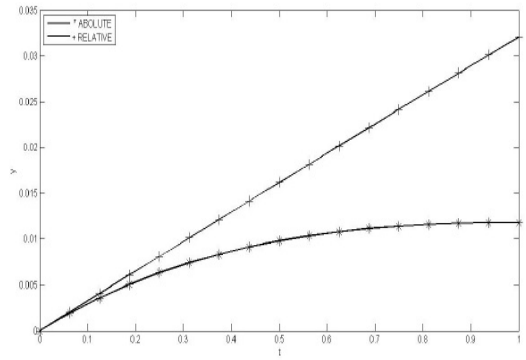

Figure 11.Absolute error and relative error

Experimental results with respect to error: From the derivation of theorem 5.1, it is observed that, the local and global truncation errors get increased from time steps $t = t_0 = a$ to $t = t_1$ and upto $t = t_N = b$ since the numerical method (2) is a recurrence relation between two consecutive time steps $t = t_i$ and $t = t_{i+1}$ the error in time steps $t = t_i$ get added with the error in the step $t = t_{i+1}$. Similarly, absolute and relative errors get increased from time steps $t = t_0$ to $t = t_1$ and upto $t = t_N = b$. This is illustrated in table 3.

It is observed from table 3 that the value of the absolute error and cummulative absolute error increases from the nodal point i = 0 to i = 16 and reaches maximum values 1.1805E-02 and 1.4227E-01 respectively. Similarly, the value of the relative error and cummulative relative error increases from the nodal point i = 0 to i =16 and reaches maximum values 3.2090E-02 and 2.7416E-01 respectively. Accummulation of error at each nodal points causes this effect.

Table 3.Exact and numerical solutions, absolute and relative errors

$t_i = h.i$	$y(t_i)$	y_i	$e_i = y(t_i)$ - y_i	$\sum_{j=0}^{i} e_j$	$r_i = 1 - \frac{y_i}{y(t_i)}$	$\sum_{j=0}^{i} r_j$
2^{-4}.0	1.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2^{-4}.1	9.3941E-01	9.3750E-01	1.9131E-03	1.9131E-03	2.0364E-03	2.0364E-03
2^{-4}.2	8.8250E-01	8.7891E-01	3.5907E-03	5.5037E-03	4.0687E-03	6.1052E-03
2^{-4}.3	8.2903E-01	8.2397E-01	5.0545E-03	1.0558E-02	6.0969E-03	1.2202E-02
2^{-4}.4	7.7880E-01	7.7248E-01	6.3246E-03	1.6883E-02	8.1209E-03	2.0323E-02
2^{-4}.5	7.3162E-01	7.2420E-01	7.4192E-03	2.4302E-02	1.0141E-02	3.0464E-02
2^{-4}.6	6.8729E-01	6.7893E-01	8.3551E-03	3.2657E-02	1.2157E-02	4.2620E-02
2^{-4}.7	6.4565E-01	6.3650E-01	9.1478E-03	4.1805E-02	1.4168E-02	5.6789E-02
2^{-4}.8	6.0653E-01	5.9672E-01	9.8112E-03	5.1616E-02	1.6176E-02	7.2965E-02
2^{-4}.9	5.6978E-01	5.5942E-01	1.0358E-02	6.1974E-02	1.8179E-02	9.1144E-02
2^{-4}.10	5.3526E-01	5.2446E-01	1.0801E-02	7.2775E-02	2.0179E-02	1.1132E-01
2^{-4}.11	5.0283E-01	4.9168E-01	1.1150E-02	8.3925E-02	2.2174E-02	1.3350E-01
2^{-4}.12	4.7237E-01	4.6095E-01	1.1415E-02	9.5340E-02	2.4165E-02	1.5766E-01
2^{-4}.13	4.4375E-01	4.3214E-01	1.1605E-02	1.0695E-01	2.6153E-02	1.8382E-01
2^{-4}.14	4.1686E-01	4.0513E-01	1.1729E-02	1.1867E-01	2.8136E-02	2.1195E-01
2^{-4}.15	3.9161E-01	3.7981E-01	1.1793E-02	1.3047E-01	3.0115E-02	2.4207E-01
2^{-4}.16	3.6788E-01	3.5607E-01	1.1805E-02	1.4227E-01	3.2090E-02	2.7416E-01

Experimental results with respect to percentage: The ratio of nunerical and exact solution $s_i = \frac{y_i}{y(t_i)}$ and its percentage is calculated. The expected ratio value is 1 at all nodal points and its percentage is 100 at all nodal points. And, the percentage of relative error is zero at all nodal points. The sum of the ratio percentage and the ralative error percentage at each nodal point is 100.

It is observed from table 4 that at the nodal point i = 0, the ratio is 1 and for i = 1 to i = 16 ratio gets decreased to the minimum value at i = 16 as 0.96791. The percentage of the ratio is 100 at i = 0 and gets decresed to the minimum 96.791 at i = 16. Similarly, the the percentage of relative error is 0 at i-0 and increased to 3.209 at i = 16. But, the sum of the ratio percentage and the ralative error percentage at each nodal point is 100.

Table 4.Exact and numerical solutions, percentage of y_i and relative errors

$t_i = h.i$	$y(t_i)$	y_i	$s_i = \frac{y_i}{y(t_i)}$	$s_i.100$	$r_i = 1 - \frac{y_i}{y(t_i)}$	$r_i.100$
$2^{-4}.0$	1.0000E+00	1.0000E+00	0.0000E+00	100.00	0.0000E+00	0
$2^{-4}.1$	9.3941E-01	9.3750E-01	9.3750E-01	99.796	2.0364E-03	0.20364
$2^{-4}.2$	8.8250E-01	8.7891E-01	9.9593E-01	99.593	4.0687E-03	0.40687
$2^{-4}.3$	8.2903E-01	8.2397E-01	9.9390E-01	99.390	6.0969E-03	0.60969
$2^{-4}.4$	7.7880E-01	7.7248E-01	9.9188E-01	99.880	8.1209E-03	0.81209
$2^{-4}.5$	7.3162E-01	7.2420E-01	9.8986E-01	98.986	1.0141E-02	1.0141
$2^{-4}.6$	6.8729E-01	6.7893E-01	9.8784E-01	98.784	1.2157E-02	1.2157
$2^{-4}.7$	6.4565E-01	6.3650E-01	9.8583E-01	98.583	1.4168E-02	1.4168
$2^{-4}.8$	6.0653E-01	5.9672E-01	9.8382E-01	98.382	1.6176E-02	1.6176
$2^{-4}.9$	5.6978E-01	5.5942E-01	9.8182E-01	98.820	1.8179E-02	1.8179
$2^{-4}.10$	5.3526E-01	5.2446E-01	9.7982E-01	97.820	2.0179E-02	2.0179
$2^{-4}.11$	5.0283E-01	4.9168E-01	9.7783E-01	97.783	2.2174E-02	2.2174
$2^{-4}.12$	4.7237E-01	4.6095E-01	9.7583E-01	97.583	2.4165E-02	2.4165
$2^{-4}.13$	4.4375E-01	4.3214E-01	9.7385E-01	97.385	2.6153E-02	2.6153
$2^{-4}.14$	4.1686E-01	4.0513E-01	9.7186E-01	97.186	2.8136E-02	2.8136
$2^{-4}.15$	3.9161E-01	3.7981E-01	9.6988E-01	96.988	3.0115E-02	3.0115
$2^{-4}.16$	3.6788E-01	3.5607E-01	9.6791E-01	96.791	3.2090E-02	3.2090

8. CONCLUSION

Calculating the shape of the unknown curve which starts at a given point and satisfies the given stiff differential equation is our problem. The shape of the unknown curve is obtained by Euler's method with order of convergence. Theoretical and numerical rate of order of convergence are derived and applied to stiff problems. Stability regions, order star, order star finger region, relative stability region, relative absolute region and order graph are presented for stiff problem. We hope that the researchers will get in deep of the development of numerical methods in the path of development of stability region in complex plane, percentage of numerical solution accuracy and theoretical and numerical rate and average rate of order convergence. It must be noted that the method presented in this paper is not A-stable and L-stable.

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