

## RESEARCH ARTICLE

# A Distribution-Free Model with Variable Setup Cost, Backorder Price Discount and Controllable Lead Time

\*Asif Iqbal Malik<sup>1</sup>, Biswajit Sarkar<sup>1</sup>

<sup>1</sup>Department of Industrial & Management Engineering, Hanyang University, Ansan Gyeonggi-do, 15588, South Korea.

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## ABSTRACT

The purpose of this study is to investigate an inventory model for variable setup cost under stochastic conditions, in which the order quantity, reorder point, lead time and backorder price-discount rate are decision variables. The lead time demand is stochastic and exact distribution is unknown. Therefore, the distribution-free approach for the lead time demand with known mean and standard deviation is discussed. The proposed method is validated based on the numerical example and sensitivity analysis, and the results are compared with the existing studies in literature. The numerical results demonstrate that the proposed distribution free approach is more economical and saves significant cost.

**Keywords:** Inventory management, Setup cost reduction, Stochastic condition, Controllable lead time, Distribution-free approach.

## 1. INTRODUCTION

In this competitive business environment, one of the very best ways to attract customers is by providing better service with shorter lead time. The process of decreasing lead time at an additional investment is known as lead time reduction. If a supplier provides better service than others, then it may increase the chances of receiving orders in future. Reduction in lead time can decrease the lost sales due to stock out, and better services improve customer's satisfaction level. There are many researchers who consider the lead time as constant or fixed. But in many real life situations, lead time is not fixed and it may vary with conditions. Hence, researchers should consider lead time as a variable in inventory problems. In practice, the distributional information about the demand is often limited. Therefore, it is a challenge to the managers to make a decision without having an idea of the distribution of the lead time demand [1]. In order to solve this problem, [2] first established the solution of a distribution free newsboy problem with known mean and standard deviation of lead time demand. [3] derived an easy proof of Scarf's [2] ordering rule.

[4] developed a probabilistic inventory model with a variable lead time. [5] assumed a continuous review inventory model with both lead time and order quantity as decision variables. [6] generalized the model as in [5] by considering shortages with partial backorders. [7] introduced inventory model with backorders and lost sale by considering mixtures of distributions for lead time demand. [8] considered a mixture inventory model with lead time and order quantity as decision variables. [9] extended the model as in [6] by assuming the reorder point as one of the decision variables and utilized distribution free approach. [10] assumed a mixed inventory model with variable lead time when the received amount is uncertain. [11] developed a continuous review inventory model with controllable lead time and learning effect. In their model, they divided lead time into three main components: setup time, productive time, and non-production time. [12] developed a mixed inventory problem with both lead time and order quantity as decision variables. Further, [13] considered a min-max distribution free procedure for an integrated inventory model with defective goods and stochastic lead time demand. [14] developed a lead time reduction model under order crossover. They examined the case where lead times are exponential in nature.

[15] developed a continuous review inventory model with a fill rate and a negative exponential crashing cost function under controllable lead time. [16] developed an integrated inventory model with variable lead time,

\*Corresponding author. Tel.: +821029541512

Email address: [ai.asifqbalmalik@gmail.com](mailto:ai.asifqbalmalik@gmail.com) (A.I.Malik)

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defective units, and delay-in-payments. A continuous review inventory model is proposed in [17] with shortages, in which an uncertain quantity is received. Also the lead time crashing cost is defined as an exponential function of lead time, and order processing cost and the lost sales rate are given by the logarithmic function of capital investment. In existing literature, researchers studied lead time reduction with the fixed or constant setup cost. The total inventory costs are reducible with the reduction in setup cost and more profit can be gained. For the first time in literature, [18] introduced a concept of an additional investment for setup cost reduction in inventory models. Similarly [6] developed a model depending on lead time and ordering cost reduction. [19] The curve effect of similar model on cost setup reduction that includes the controllable lead time, mixture of backorder and partial lost sales was discussed. [20] A periodic review inventory model was derived based on controllable setup cost and lead time. [21] investigated an inventory model to show the setup cost reduction on controllable backorder rate and variable lead time. [22] studied an integrated inventory model with setup cost reduction for vendors.

Generally, stock out situations occur whenever supplier does not have sufficient stock to fulfill the customer's order. In this situation, the supplier may miss the opportunity to earn more profit and disappoint customers. In assumptions of basic inventory model, the shortages are either completely backordered or completely lost. But in practical situation, some customers may wait for backorders for their items, while others may refuse to wait for the backorders. One of the factors to convince customers for backorders is price discount offers to customers over backordered items. [23] presented an EOQ model for backorder price-discount with variable lead time. [7] developed an inventory model for variable lead time demand with mixture of backorders and lost sales. [24] studied a (Q,r) inventory model with a mixture of backorders and lost sales during stock out period in which the lead time, order quantity and backorder discount are considered as decision variables. [25] investigated a vendor-buyer inventory model with variable lead time and backorder discount. [26] presented an inventory model with negative exponential backorder rate and service level constraint. [27] considered an inventory model with backorder price-discounts and variable lead time demand. In the same direction, [28] presented a distribution-free inventory model with backorder price-discount, controllable lead time and reduction in ordering cost. Further, [29] a continuous review model depending on backorder price-discount and variable lead time is investigated for the purpose of increasing investment and reducing the annual total cost effectively.

Generally, authors considered continuous review inventory model with normally distributed lead time demand and fixed setup cost. Also, most of the previously mentioned studies considered lead time as a fixed parameter. Therefore, to fulfill the existing research gap, this study investigates a stochastic continuous review inventory model with controllable lead time. The setup cost, reorder point, order quantity, backorder price discount and lead time are considered as decision variables. Further, distribution free approach for the lead time demand is considered. In earlier studies, mostly authors considered normally distributed lead time demand but in real situations, this is not possible. The goal of this research is to minimize the total related cost and optimize the setup cost, order quantity, backorder price discount, reorder point and lead time, simultaneously. For solution, an efficient computational algorithm is presented. In this paper, notation and assumptions are given in the next section. The mathematical model is presented in section 3. In section 4, the numerical example and sensitivity analysis is stated. Finally, conclusions and discussion are given in section 5.

## 2. NOTATION AND ASSUMPTIONS

We use the following notation to develop the model.

### Decision variables

- Q     order quantity (units)
- A     setup cost per setup, after investment for setup cost reduction (\$/setup)
- L     length of the lead time (days)
- $\pi_x$    backorder price-discount per unit offered by supplier (\$/unit)
- k     safety factor

### Parameters

- D     annual average demand (units/year)
- $A_0$    initial setup cost per production, before investment for setup cost reduction (\$/setup)
- h     inventory \* holding cost per unit per year (\$/unit/year)
- $\pi_0$    marginal profit per unit (\$/unit)
- $\alpha$    annual fractional cost of the capital investment (\$/year)
- C(L) lead time crashing cost function

- $\beta$  backorder ratio
- $\gamma$  reorder point
- $\beta_0$  upper bound of the backorder ratio
- $\sigma$  standard deviation of the lead time demand
- $a_i$   $i^{th}$  component of lead time with  $u_i$  as minimum duration (days),  $i=1, 2, \dots, n$
- $b_i$   $i^{th}$  component of lead time with  $v_i$  as normal duration (day),  $i=1, 2, \dots, n$
- $c_i$   $i^{th}$  component of lead time with  $m_i$  as crashing cost per day,  $i=1, 2, \dots, n$
- $X$  lead time demand, which has a distribution function  $F$  with finite mean  $DL$  and standard deviation  $\sigma\sqrt{L}$
- $E(x)$  mathematical expectation of  $x$
- $x^+$   $\max \{x, 0\}$
- $E(X - r)^+$  expected shortage quantity at the end of the cycle.

The following assumptions are considered to develop the model.

1. A single type continuously reviewed inventory model with setup cost is considered. When inventory level falls to reorder point  $r$ , replenishment occurs.
2. Determination of reorder point is done by  $r = DL + k\sigma\sqrt{L}$ , where  $DL =$  expected demand during the lead time, and  $k\sigma\sqrt{L} =$  safety stock.
3. Additional investment function possibly reduces the setup cost. Therefore, a continuous investment function is assumed in which the setup cost is reduced by logarithmic investment function, which is convex with restriction  $0 < A \leq A_0$  for all  $A$ .
4. We suppose  $L_0 = \sum_{j=1}^n b_j$  and  $L_i$  as [21].

$$L_i = \sum_{j=1}^n b_j - \sum_{j=1}^i (b_j - a_j)$$

where  $i = 1, 2, \dots, n$ . The lead time crashing cost function  $C(L)$  per cycle is given as,

$$C(L) = m_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$$

5. Consider the backorder ratio ( $\beta$ ) as a variable, which is proportioned to the price-discount presented by the supplier per unit  $\pi_x$ . Hence,  $\beta = \beta_0\pi_x/\pi_0$  with restriction  $0 \leq \pi_x \leq \pi_0$  and  $0 \leq \beta_0 \leq 1$ .

### 3. MATHEMATICAL MODEL

Generally, lead time demand is considered as the normally distributed, however, in many practical life problems, the information known about the probability distribution of the lead time is limited. In this model, it is only assumed that the density function of the lead time demand belongs to  $\Omega$  with finite mean  $DL$  and standard deviation  $\sigma\sqrt{L}$ . As the distributional form of lead time demand  $X$  is unknown, the exact value of  $E(X - r)^+$  cannot be determined. Therefore, the min-max distribution free approach is considered to solve this problem [2].

$$\text{MinMax}_{F \in \Omega} \text{TEC}(Q, A, \pi_x, k, L)$$

subject to

$$0 < A < A_0$$

$$0 < \pi_x < \pi_0$$

In this model, the Gallego and Moon's [2] proposition is used directly to approximate the value of  $E(X - r)^+$ .

#### Proposition 1

For any  $F \in \Omega$ ,

$$E(X - r)^+ \leq \frac{1}{2} \sqrt{\sigma^2 L + (r - DL)^2 - (r - DL)}$$

According to [2], the upper bound is tight.

Substituting  $r = DL + k\sigma\sqrt{L}$  into above equation, we can obtain,

$$E(X - r)^+ \leq \frac{1}{2} \sigma\sqrt{L}(\sqrt{1+k^2} - k)$$

Thus, the safety factor  $k$  and the backorder price discount  $\pi_x$  offered by the supplier per unit can be treated as decision variables.

[17] utilises an initial logarithmic investments function  $I_A(A)$  for setup cost reduction and is given as,

$$I_A(A) = B \ln\left(\frac{A_0}{A}\right), \quad 0 < A \leq A_0 \tag{1}$$

where  $B = \frac{1}{\delta}$  and  $\delta$  =percentage decrease in  $A$  per dollar increase in  $I_A(A)$ . If the limit constraints  $A \in (0, A_0]$  in above equation is not satisfied, then setup cost cannot be reduced. In such a case, optimal setup cost is considered as the original setup cost.

The expected annual total cost is conveyed as,

$$TEC(Q, A, \pi_x, r, k, L) = \text{Setup cost} + \text{Investment for setupcost reduction} + \text{holding cost} + \text{stock out cost} + \text{lead time crashing cost} \tag{2}$$

$$TEC(Q, A, \pi_x, r, k, L) = \frac{AD}{Q} + \alpha\beta \ln\left(\frac{A_0}{A}\right) + h \left[ \frac{Q}{2} + r - DL + (1 - \beta) \frac{1}{2} \sigma\sqrt{L}(\sqrt{1+k^2} - k) \right] + \frac{D}{Q} [\pi_x\beta - \pi_0(1 - \beta)] \frac{1}{2} \sigma\sqrt{L}(\sqrt{1+k^2} - k) + \frac{D}{Q} C(L)$$

Substituting  $r = DL + k\sigma\sqrt{L}$  and  $\beta = \beta_0\pi_x/\pi_0$  in (1), we obtain,

$$TEC(Q, A, \pi_x, k, L) = \frac{AD}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L} \right] + \frac{1}{2} \left[ h \left( 1 - \frac{\pi_x\beta_0}{\pi_0} \right) + \frac{D\bar{\pi}}{Q} \right] \frac{1}{2} \sigma\sqrt{L}(\sqrt{1+k^2} - k) + \frac{D}{Q} C(L) \tag{3}$$

where  $\bar{\pi} = \pi_0 - \beta_0\pi_x + \beta_0\pi_x^2/\pi_0$

$$TEC(Q, A, \pi_x, k, L) = \alpha\beta \ln\left(\frac{A_0}{A}\right) + \frac{AD}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L} + \left( 1 - \frac{\beta_0\pi_x}{\pi_0} \right) \sigma\sqrt{L}(\sqrt{1+k^2} - k) \right] + \frac{D}{Q} \left[ \frac{\beta_0\pi_x^2}{\pi_0} + \pi_0 - \beta_0\pi_x \right] \sigma\sqrt{L}(\sqrt{1+k^2} - k) + \frac{D}{Q} C(L) \tag{4}$$

Thus, the aim is to minimize TEC regarding five decision variables and two different constraints.

$$\text{Min } TEC(Q, A, \pi_x, k, L) = \alpha\beta \ln\left(\frac{A_0}{A}\right) + \frac{AD}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L} + \left( 1 - \frac{\beta_0\pi_x}{\pi_0} \right) \sigma\sqrt{L}(\sqrt{1+k^2} - k) \right] + \frac{D}{Q} \left[ \frac{\beta_0\pi_x^2}{\pi_0} + \pi_0 - \beta_0\pi_x \right] \sigma\sqrt{L}(\sqrt{1+k^2} - k) + \frac{D}{Q} C(L) \tag{5}$$

subject to restrictions,

$$0 < A \leq A_0$$

$$0 \leq \pi_x \leq \pi_0$$

Non linear problem exists, which can be solved by relaxing the constraints  $0 < A \leq A_0$  and  $0 \leq \pi_x \leq \pi_0$ . The first order partial derivatives of  $TEC(Q, A, \pi_x, k, L)$  based on  $A, Q, k, \pi_x$  and  $L$  is calculated as,

$$\frac{\partial TEC}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{D\bar{\pi}\sigma\sqrt{L}}{2Q^2} (\sqrt{1+k^2} - k) - \frac{D}{Q^2} C(L) \tag{6}$$

$$\frac{\partial TEC}{\partial k} = h\sigma\sqrt{L} - \left[ h \left( 1 - \frac{\beta_0\pi_x}{\pi_0} \right) + \frac{D}{Q}\bar{\pi} \right] \sigma\sqrt{L} \left( \frac{k}{\sqrt{1+k^2}} - 1 \right) \tag{7}$$

$$\frac{\partial TEC}{\partial \pi_x} = \left[ \frac{D}{Q} \left( \frac{2\beta_0\pi_x}{\pi_0} - \beta_0 \right) - \frac{h\beta_0}{\pi_0} \right] \sigma\sqrt{L}(\sqrt{1+k^2} - k) \tag{8}$$

$$\frac{\partial TEC}{\partial A} = -\frac{\alpha\beta}{A} + \frac{D}{Q} \tag{9}$$

$$\frac{\partial TEC}{\partial L} = \frac{hk\sigma}{2\sqrt{L}} + \frac{\sigma(\sqrt{1+k^2} - k)}{2\sqrt{L}} \left[ h \left( 1 - \frac{\beta_0\pi_x}{\pi_0} \right) + \frac{D}{Q}\bar{\pi} \right] - \frac{m_i D}{Q} \tag{10}$$

For the fixed Q, A, k and  $\pi_x$ ,  $TEC(Q, A, \pi_x, k, L)$  is concave in L,

$$\frac{\partial^2 TEC}{\partial L^2} = -\frac{1}{4}hk\sigma L^{-3/2} - \frac{1}{4} \left[ h \left( 1 - \frac{\beta_0\pi_x}{\pi_0} \right) + \frac{D}{Q}\bar{\pi} \right] \sigma L^{-3/2} (\sqrt{1+k^2} - k)$$

Hence,

$$\frac{\partial^2 TEC}{\partial L^2} < 0$$

Thus, the minimum total expected annual cost occurs at the internal end points  $[L_i, L_{i-1}]$  for fixed (Q, A, k,  $\pi_x$ ) and is obtained by equating (5)-(8) to zero,

$$Q = \sqrt{\frac{D \left[ 2A + \bar{\pi}\sigma\sqrt{L}(\sqrt{1+k^2} - k) \right] 2C(L)}{h}} \tag{11}$$

$$\frac{k}{\sqrt{1+k^2}} = 1 - \frac{2Qh}{hQ \left( 1 - \frac{\beta_0\pi_x}{\pi_0} \right) + D\bar{\pi}} \tag{12}$$

$$\pi_x = \frac{hQ}{2D} + \frac{\pi_0}{2} \tag{13}$$

$$A = \frac{\alpha BQ}{D} \tag{14}$$

For fixed  $L \in [L_i, L_{i-1}]$ , when the constraints  $0 < A \leq A_0$  and  $0 \leq \pi_x \leq \pi_0$  the near optimal values of Q, k,  $\pi_x$ , A (we denoted as  $Q^{**}, k^{**}, \pi_x^{**}, A^{**}$ ) are obtained by ignoring in which a minimum annual cost is expected.

**Preposition 2**

For fixed  $L \in [L_i, L_{i-1}]$  the Hessian matrix of  $TEAC(Q, A, \pi_x, k, L)$  is positive at the point  $(Q^{**}, k^{**}, \pi_x^{**}, A^{**})$  as obtained from (10)-(13).

**Proof**

For given value of L, the Hessian matrix H is,

$$H = \begin{bmatrix} \frac{\partial^2 TEC(\cdot)}{\partial Q^{**2}} & \frac{\partial^2 TEC(\cdot)}{\partial Q^{**}\partial \pi_x^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial Q^{**}\partial k^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial Q^{**}\partial A^{**}} \\ \frac{\partial^2 TEC(\cdot)}{\partial \pi_x^{**}\partial Q^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial \pi_x^{**2}} & \frac{\partial^2 TEC(\cdot)}{\partial \pi_x^{**}\partial k^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial \pi_x^{**}\partial A^{**}} \\ \frac{\partial^2 TEC(\cdot)}{\partial k^{**}\partial Q^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial k^{**}\partial \pi_x^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial k^{**2}} & \frac{\partial^2 TEC(\cdot)}{\partial k^{**}\partial A^{**}} \\ \frac{\partial^2 TEC(\cdot)}{\partial A^{**}\partial Q^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial A^{**}\partial \pi_x^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial A^{**}\partial k^{**}} & \frac{\partial^2 TEC(\cdot)}{\partial A^{**2}} \end{bmatrix}$$

where  $TEAC(\cdot) = TEC(Q^{**}, K^{**}, \pi_x^{**}, L^{**} A^*)$

$$\begin{aligned} \frac{\partial^2 TEC(\cdot)}{\partial Q^{**2}} &= \frac{2A^{**}D}{Q^{**3}} + \frac{2D\bar{\pi}\sigma\sqrt{L}(\sqrt{1+K^{**2}}-k^{**})}{Q^{**3}} + \frac{2DC(L)}{Q^{**3}} \\ \frac{\partial^2 TEC(\cdot)}{\partial \pi_x^{**2}} &= \frac{2D}{Q^{**}} \left( \frac{\beta_0\pi_x^{**}}{\pi_0} - \beta_0 \right) \sigma\sqrt{L}(\sqrt{1+K^{**2}}-k^{**}) \\ \frac{\partial^2 TEC(\cdot)}{\partial K^{**2}} &= \left[ \frac{D}{Q^{**}}\bar{\pi} + h \left( 1 - \frac{\beta_0\pi_x^{**}}{\pi_0} \right) \right] \sigma\sqrt{L}(\sqrt{1+K^{**2}}-k^{**}) \\ \frac{\partial^2 TEC(\cdot)}{\partial A^{**2}} &= \frac{\alpha B}{A^{**2}} \\ \frac{\partial^2 TEC(\cdot)}{\partial Q^{**} \partial \pi_x^{**}} &= \frac{\partial^2 TEC(\cdot)}{\partial \pi_x^{**} \partial Q^{**}} = \frac{D\sigma\sqrt{L}(\sqrt{1+K^{**2}}-k^{**})}{Q^{**2}} \left( \beta_0 - \frac{2\beta_0\pi_x^{**}}{\pi_0} \right) \\ \frac{\partial^2 TEC(\cdot)}{\partial Q^{**} \partial k^{**}} &= \frac{\partial^2 TEC(\cdot)}{\partial k^{**} \partial Q^{**}} = \frac{D\bar{\pi}\sigma\sqrt{L}}{Q^{**2}} (\sqrt{1+K^{**2}}-k^{**}) \\ \frac{\partial^2 TEC(\cdot)}{\partial Q^{**} \partial A^{**}} &= \frac{\partial^2 TEC(\cdot)}{\partial A^{**} \partial Q^{**}} = -\frac{D}{Q^{**2}} \\ \frac{\partial^2 TEC(\cdot)}{\partial k^{**} \partial \pi_x^{**}} &= \frac{\partial^2 TEC(\cdot)}{\partial \pi_x^{**} \partial k^{**}} = \left[ \frac{h\beta_0}{\pi_0} - \frac{D}{Q^{**}} \left( \frac{2\beta_0\pi_x^{**}}{\pi_0} - \beta_0 \right) \right] \sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**}) \\ \frac{\partial^2 TEC(\cdot)}{\partial k^{**} \partial A^{**}} &= \frac{\partial^2 TEC(\cdot)}{\partial A^{**} \partial k^{**}} = 0 \\ \frac{\partial^2 TEC(\cdot)}{\partial \pi_x^{**} \partial A^{**}} &= \frac{\partial^2 TEC(\cdot)}{\partial A^{**} \partial \pi_x^{**}} = 0 \end{aligned}$$

The first principal minor of H is,

$$|H_{11}| = \frac{2A^{**}D}{Q^{**3}} + \frac{2D\bar{\pi}\sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**})}{Q^{**3}} + \frac{2DC(L)}{Q^{**3}} > 0$$

The second principal minor of H is,

$$\begin{aligned} |H_{22}| &= \left[ \frac{2A^{**}D}{Q^{**3}} + \frac{2D\bar{\pi}\sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**})}{Q^{**3}} + \frac{2DC(L)}{Q^{**3}} \right] \left[ \frac{2D\beta_0}{Q^{**}\pi_0} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**}) \right] \\ &\quad - \left\{ \frac{D^2}{Q^{**4}} \left( \frac{2\beta_0\pi_x^{**}}{\pi_0} + \beta_0 \right)^2 \left[ \sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**}) \right]^2 \right\} \\ &= \frac{4D^2\beta_0\sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**})}{Q^{**4}\pi_0} [A^{**} + C(L)] + \frac{D^2\beta_0[\sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**})]^2}{Q^{**4}} (4-\beta_0) > 0 \end{aligned}$$

The third principal minor of H is,

$$\begin{aligned} |H_{33}| &= \left\{ \frac{4D^2\beta_0\sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**})}{Q^{**4}\pi_0} [A^{**} + C(L)] + \frac{D^2\beta_0[\sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**})]^2}{Q^{**4}} (4-\beta_0) \right\} \\ &\quad \left[ \frac{D}{Q^{**}}\bar{\pi} + h \left( 1 - \frac{\beta_0\pi_x^{**}}{\pi_0} \right) \right] \sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**}) - \frac{2D\beta_0}{Q^{**}\pi_0} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-K^{**}) \left[ \frac{D\bar{\pi}\sigma\sqrt{L}}{Q^{**2}} (\sqrt{1+k^{**2}}-K^{**}) \right]^2 \\ &= \frac{4D^2\beta_0\sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})}{Q^{**4}\pi_0} [A^{**} + C(L)] \left\{ \frac{D}{Q^{**}}\bar{\pi} + h \left( 1 - \frac{\beta_0\pi_x^{**}}{\pi_0} \right) \right\} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) + \end{aligned}$$

$$\begin{aligned}
 & \frac{D^2\beta_0 \left\{ \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) \right\}^2}{Q^{**4}} (4-\beta_0)h \left( 1 - \frac{\beta_0\pi_x^{**}}{\pi_0} \right) \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) + \frac{D^2\beta_0 \left\{ \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) \right\}^2}{Q^{**4}} \\
 & (4-\beta_0) \frac{D}{Q^{**}} \bar{\pi} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) - \frac{2D\beta_0}{Q^{**}\pi_0} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) \left[ \frac{D^2\bar{\pi}^2\sigma^2L}{Q^{**4}} \left\{ (\sqrt{1+k^{**2}}-k^{**}) \right\}^2 \right] \\
 & > \frac{D^3\beta_0 \left\{ \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) \right\}^2}{Q^{**5}} (4-\beta_0)\bar{\pi}\sigma(\sqrt{1+k^{**2}}-k^{**}) \\
 & \quad - \frac{\beta_0\sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})}{\pi_0} \left[ \frac{2D^3\bar{\pi}^2\sigma^2L}{Q^{**5}} \left\{ (\sqrt{1+k^{**2}}-k^{**}) \right\}^2 \right] \\
 = & \frac{D^3\sigma^2L}{Q^{**5}} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})\bar{\pi} \left[ \beta_0(\sqrt{1+k^{**2}}-k^{**})(4-\beta_0)(\sqrt{1+k^{**2}}-k^{**}) - 2\frac{\beta_0}{\pi_0} \left\{ (\sqrt{1+k^{**2}}-k^{**}) \right\}^2 \bar{\pi} \right] \\
 = & \frac{D^3\sigma^2L}{Q^{**5}} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})\bar{\pi} \left[ \beta_0(\sqrt{1+k^{**2}}-k^{**})(4-\beta_0)(\sqrt{1+k^{**2}}-k^{**}) \right. \\
 & \quad \left. - 2 \left( (\sqrt{1+k^{**2}}-k^{**}) \right)^2 \left\{ \frac{\beta_0^2\pi_x^{**}}{\pi_0^2} (\pi_x^{**} - \pi_0) + \beta_0 \right\} \right] \\
 > & \frac{D^3\sigma^2L}{Q^{**5}} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})\bar{\pi} \left[ \beta_0(\sqrt{1+k^{**2}}-k^{**})(4-\beta_0)\phi(k^*) - 2\beta_0 \left( (\sqrt{1+k^{**2}}-k^{**}) \right)^2 \right] \\
 = & \frac{D^3\sigma^2L}{Q^{**5}} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})\bar{\pi} \left[ \beta_0(2-\beta_0)(\sqrt{1+k^{**2}}-k^{**}) + 2\beta_0(\sqrt{1+k^{**2}}-k^{**}) \right. \\
 & \quad \left. - 2\beta_0 \left( (\sqrt{1+k^{**2}}-k^{**}) \right)^2 \right] \\
 > & \frac{2D^3\sigma^2L\beta_0}{Q^{**5}} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})\bar{\pi} \left[ \Psi(k^*)(\sqrt{1+k^{**2}}-k^{**}) - \left( (\sqrt{1+k^{**2}}-k^{**}) \right)^2 \right] > 0
 \end{aligned}$$

The fourth principal minor determinant of H is,

$$\begin{aligned}
 |H_{44}| = & \frac{\alpha B}{A^{**2}} |H_{33}| + \frac{D}{Q^{**2}} \left[ - \frac{2D\beta_0\sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})}{Q^{**}\pi_0} \frac{D}{Q^{**2}} \left\{ \frac{D\bar{\pi}}{Q^{**}} + h \left( 1 - \frac{\beta_0\pi_x^{**}}{\pi_0} \right) \right\} \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) \right. \\
 & \left. + \left\{ \frac{h\beta_0}{\pi_0} - \frac{D}{Q^{**}} \left( \frac{2\beta_0\pi_x^{**}}{\pi_0} - \beta_0 \right) \right\}^2 \left( \sigma\sqrt{L}[(\sqrt{1+k^{**2}}-k^{**})] \right)^2 \frac{D}{Q^{**2}} \right] \\
 = & \frac{\alpha B}{A^{**2}} |H_{33}| + \left( \frac{D}{Q^{**2}} \right)^2 \left\{ \frac{h\beta_0}{\pi_0} - \frac{D}{Q^{**}} \left( \frac{2\beta_0\pi_x^{**}}{\pi_0} - \beta_0 \right) \right\}^2 \left( \sigma\sqrt{L}[(\sqrt{1+k^{**2}}-k^{**})] \right)^2 \\
 - & \left( \frac{D}{Q^{**2}} \right)^2 \left[ \frac{2D\beta_0\sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})}{Q^{**}\pi_0} \left\{ \frac{D\bar{\pi}}{Q^{**}} + h \left( 1 - \frac{\beta_0\pi_x^{**}}{\pi_0} \right) \right\} \right] \sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**}) \\
 = & \frac{\alpha B}{A^{**2}} |H_{33}| + \left( \frac{D}{Q^{**2}} \right)^2 \left[ \left\{ \frac{h\beta_0}{\pi_0} - \frac{D}{Q^{**}} \left( \frac{2\beta_0\pi_x^{**}}{\pi_0} - \beta_0 \right) \right\}^2 \left( \sigma\sqrt{L}[(\sqrt{1+k^{**2}}-k^{**})] \right)^2 \right. \\
 & \left. - \frac{2D\beta_0\sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})\sigma\sqrt{L}(\sqrt{1+k^{**2}}-k^{**})}{Q^{**}\pi_0} \left\{ \frac{D\bar{\pi}}{Q^{**}} + h \left( 1 - \frac{\beta_0\pi_x^{**}}{\pi_0} \right) \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\alpha B}{A^{**2}} |H_{33}| + \left( \frac{D}{Q^{**2}} \right)^2 \left[ \left\{ \frac{h\beta_0}{\pi_0} - \frac{D\beta_0}{Q^{**}\pi_0} (2\pi_x^{**} - \beta_0) \right\}^2 \left( \sigma\sqrt{L}[(\sqrt{1+k^{**2}} - k^{**})] \right)^2 \right. \\
 &\quad \left. - \frac{2D\beta_0\sigma^2L(\sqrt{1+k^{**2}} - k^{**})h}{Q^{**}\pi_0(\sqrt{1+k^{**2}} - k^{**})} \right] \\
 &= \frac{\alpha B}{A^{**2}} |H_{33}| + \left( \frac{D}{Q^{**2}} \right)^2 \left[ \left\{ \frac{\beta_0}{\pi_0 Q^{**}} \left( hQ^{**} - 2D \left( \pi_x^{**} - \frac{\pi_0}{2} \right) \right) \right\}^2 \left( \sigma\sqrt{L}[(\sqrt{1+k^{**2}} - k^{**})] \right)^2 \right. \\
 &\quad \left. - \frac{2D\beta_0\sigma^2L(\sqrt{1+k^{**2}} - k^{**})h}{Q^{**}\pi_0(\sqrt{1+k^{**2}} - k^{**})h} \right] \\
 &> \frac{\alpha B}{A^{**2}} |H_{33}| - \left( \frac{D}{Q^{**2}} \right)^2 \frac{2D\beta_0\sigma^2L(\sqrt{1+k^{**2}} - k^{**})h}{Q^{**}\pi_0(\sqrt{1+k^{**2}} - k^{**})} \\
 &= \frac{\alpha B}{A^{**2}} |H_{33}| + \left( \frac{D}{Q^{**2}} \right)^2 \frac{2D\beta_0\sigma^2L(\sqrt{1+k^{**2}} - k^{**})h}{Q^{**}\pi_0(\sqrt{1+k^{**2}} - k^{**})} > 0
 \end{aligned}$$

As the principal minors of the H are positive, H is positive definite at the point  $(Q^{**}, A^{**}, K^{**}, \pi_x^{**})$ . Hence the total expected annual cost function is a global minimum at that point.

Consider two constraints  $0 < A \leq A_0$  and  $0 \leq \pi_x \leq \pi_0$ . From (13) and (14), it is clear that since  $\alpha, B, Q, D, h$  and  $\pi_0$  are all positive,  $\pi_x^{**}$  and  $A^{**}$  are also said to be positive. In case of satisfying constraints  $A^{**} > A_0$  and  $\pi_x^{**} > \pi_0$ , then investment is not used for the setup cost reduction, and supplier does not offer any discount respectively. Thus it is considered to assume  $A^{**} = A_0$  and  $\pi_x^{**} = \pi_0$ . Such type of non linear problem results in the possibility of getting closed form solution with concave cost function for lead time. Also the obtained results are near optimum solutions instead of global optimum. The optimum values of variables Q, A, k,  $\pi_x$  and L is obtained by an algorithm given below.

### 3.1. Solution algorithm

**Step 1** For each  $L_i, i = 1, 2, \dots, n$ , perform 1a to 1e.

**1a** Set  $\pi_{xi1} = \pi_0, A_{i1} = A_0$  and  $k_{i1} = 0$

**1b** Substitute  $\pi_{xi1}, A_{i1}$  and  $k_{i1}$  into (11), evaluate  $Q_{i1}$ .

**1c** Utilizing  $Q_{i1}$ , obtain  $k_{i1}$  from (12).

**1d** Utilizing  $Q_{i1}$ , determine  $\pi_{xi2}$  and  $A_{i2}$  from (13) and (14), respectively.

**1e** Repeat 1a to 1d until no changes occur in the values of  $Q_i, A_i, \pi_{xi}$  and  $k_i$ . Denote the solution by  $(\widehat{Q}_i, \widehat{A}_i, \widehat{k}_i, \widehat{\pi}_{xi})$

**Step 2** For each  $i=1, 2, \dots, n$ , compare  $\widehat{\pi}_{xi}$  with  $\pi_0$  and  $\widehat{A}_i$  with  $A_0$ .

**2a** If  $\widehat{\pi}_{xi} < \pi_0$  and  $\widehat{A}_i < A_0$ , then the solution found in step 1 is the near optimal for given  $L_i$ . Denote this solution by  $(Q^{**}, K^{**}, \pi_x^{**}, A^{**})$ . Go to step 4.

**2b** If  $\widehat{\pi}_{xi} \geq \pi_0$  and  $\widehat{A}_i < A_0$ , then for given  $L_i$ , set  $\pi_{xi}^* = \pi_0$  and evaluate new  $(\widehat{Q}_i, \widehat{k}_i, \widehat{A}_i)$  from (10), (11) and (13) by the same procedure as described in step 1. If  $\widehat{A}_i < A_0$ , then the near optimal solution is  $(Q^{**}, K^{**}, \pi_x^{**}, A^*) = (\widehat{Q}_i, \widehat{k}_i, \pi_0, \widehat{A}_i)$ , and go to step 4. Otherwise, go to step 3.

**2c** If  $\widehat{A}_i \geq A_0$  and  $\widehat{\pi}_{xi} < \pi_0$ , then for given  $L_i$ , set  $A^{**} = A_0$  and evaluate new  $(\widehat{Q}_i, \widehat{k}_i, \widehat{\pi}_{xi})$  from (11), (12), and (13) by the same procedure as described in step 1. If  $\widehat{\pi}_{xi} < \pi_0$ , then the near optimal solution is  $(Q^{**}, K^{**}, \pi_x^{**}, A^{**}) = (\widehat{Q}_i, \widehat{k}_i, \pi_0, \widehat{A}_i)$ , and go to step 4. Otherwise, go to 3.

**2d** If  $\widehat{\pi}_{xi} \geq \pi_0$  and  $\widehat{A}_i \geq A_0$ , go to step 3.

**Step 3** For the given  $L_i$ , set  $\pi_{xi}^{**} = \pi_0$  and  $A_i^* = A_0$  using (11) and (12) to obtain the corresponding near optimal solution  $(Q_i^{**}, k_i^{**})$  using step 1.

**Step 4** Utilize (3) to calculate the corresponding total expected cost  $TEC(Q_i^{**}, A_i^{**}, \pi_{xi}^{**}, k_i^{**}, L_i)$ .

**Step 5**  $TEC(Q^{**}, A^{**}, \pi_x^{**}, k^{**}, L^{**}) = \min_{i=1,2,\dots,n} TEC(Q_i^{**}, A_i^{**}, \pi_{xi}^{**}, k_i^{**}, L_i)$  gives  $(Q^{**}, A^{**}, \pi_x^{**}, k^{**}, L^{**})$ , the near optimal solution. After substituting the values of  $k^{**}$  and  $L^{**}$ , the reorder point can be obtained as  $r^{**} = DL + k^{**} \sigma \sqrt{L^{**}}$



#### 4. NUMERICAL EXAMPLE

Certain parameter values are specified as follows to illustrate the presented problem as  $D = 600$  units/year,  $h = \$20$ /unit/year,  $\pi_0 = \$150$ /unit,  $A_0 = \$200$ /order,  $\sigma = 7$ units,  $\alpha = \$0.1$ /year and  $B = 5800$ , and the lead time comprises three components including data as shown in table 1.

Table 1.Lead time data

Lead time component i	Normal duration $b_i$ (days)	Minimum duration $a_i$ (days)	Unit crashing cost $c_i$ (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

**Example** The results are determined for the upper bounds of  $\beta_0 = 0.0, 0.5, 0.8, 1.0$ . Table 2 presents the near optimum results implemented using above algorithm. Table 3 provides the optimum solution obtained for distribution free model.

Table 2.The near optimal solution for normal distribution model (Li in weeks)

$\beta_0$	Investment and price discount (proposed model) $(A^{**}, Q^{**}, \pi_x^{**}, k^{**}, L^{**})$	$TEC_2(\cdot)$	No investment and no price discount( $A = A_0 = 200$ and $\pi_x = \pi_0 = 150$ ) $(Q^{**}, k^{**}, L^{**})$	$TEC_1(\cdot)$	Saving %
0.0	(143.15, 148.09, 77.47, 2.66, 3)	3824.11	(160.46, 2.55, 3)	3851.22	0.70
0.5	(140.33, 145.17, 77.42, 2.49, 3)	3731.39	(160.78, 2.53, 3)	3839.72	2.82
0.8	(138.54, 143.31, 77.39, 2.38, 3)	3671.88	(160.78, 2.51, 3)	3832.76	4.20
1.0	(137.28, 142.02, 77.37, 2.31, 3)	3630.32	(161.11, 2.50, 3)	3828.08	5.17

$$Note : Saving = \{TEC_1(\cdot) - TEC_2(\cdot) / TEC_1(\cdot)\} * 100\%$$

The impact of setup cost reduction and backorder price discount is examined by comparing the presented results with existing literature [23] and is summarised in table 3, which concludes that presented model produces better results.

Table 3.Numerical comparison with [23] model

$\beta_0$	Total cost		
	Our model	[23]	Savings %
0.0	3824.11	4163.98	08.16
0.5	3731.39	4077.86	08.50
0.8	3671.88	4016.57	08.58
1.0	3630.32	3971.22	08.58

#### 4.1. Sensitivity analysis

Studies related to effect on parameter changes such as  $A_0, h, \alpha, \sigma$  and  $B$  on total cost are performed. Sensitivity analysis is performed by changing each of the parameters by  $-50\%, -25\%, +25\%$  and  $+50\%$  taking one parameter at a time while keeping the remaining parameters unchanged. The results of example is presented in table 4.

The sensitive analysis of an important parameters are discussed as follows:

1. Total cost increases with increase in setup cost, in which an investment reduces the setup cost. Hence from table 4, it is indicated that 50% change in setup cost produces only 30% change in total cost function.
2. As the holding cost increases, the total cost also increases. Negative changes in holding cost produces greater degradation in total cost when compared to positive change. Thus holding cost becomes more sensitive to negative change related to total cost.
3. The total annual cost increases as a result of increment in annual fractional cost of capital investment.
4. If the standard deviation of lead time demand increases while all the other parameters remain unchanged, the expected total cost tends to increase. The negative and positive changes in the standard deviation give approximately the same amount of change in total cost function.
5. The total annual cost tends to increase as the value of  $B$  increases.

Table 4.Sensitivity analysis for key parameters

Parameters	Changes (in%)	Normal distribution model
$A_0$	-50%	3269.85
	-25%	3505.02
	0%	3671.88
	+25%	3801.30
	+50%	3907.05
h	-50%	2490.16
	-25%	3130.91
	0%	3671.88
	+25%	4152.13
	+50%	4590.43
$\alpha$	-50%	3454.56
	-25%	3593.39
	0%	3671.88
	+25%	3703.09
	+50%	3694.86
$\sigma$	-50%	3042.52
	-25%	3364.24
	0%	3671.88
	+25%	3967.59
	50%	4253.01
B	-50%	3454.56
	-25%	3593.39
	0%	3671.88
	+25%	3703.09
	+50%	3694.86

#### 4.2. Managerial insights

The sensitivity analysis provides some interesting managerial insights for industries. As always the exact probability distribution for stochastic demand is not known, the managers cannot decide the correct distribution function to be followed. For this problem, the proposed model suggested the distribution free approach. Setup cost is reducible at an addition investment and it significantly reduces the overall cost of the inventory system. Managers should also consider the effect of lead time on service and demand. If the lead time is longer, customers divert to other suppliers/retailers. Offering discounts over backordered quantity could be another important measure to make the inventory system more profitable.

#### 5. CONCLUSIONS

The proposed model with distribution free approach analyses the impact of setup cost reduction by considering decision variables as order quantity, backorder price discount rate, lead time and reorder point. The global optimum solution is obtained using a proposition for fixed lead time whereas this becomes impossible for a variable lead time as the conditions are dissatisfied. During shortages of stock items, price-discount policy is adopted which makes the customer to wait for ordered items thereby securing backorder from customers. From the above numerical example, it is viewed that increase in upper bound of backorder price-discount increases results in decrease of the total cost. Also it indicates that the logarithmic investment function and pre-discount policy reduces the overall system cost thereby achieving considerable savings, which further attains gain in the competitive market.

This model is applicable to conditions where stock out situations occur and customers are willing to wait for desired items with price discount. There are several possible strategies for the production managers. The managers can make decisions based on the available limited information about lead time demand distribution and consider distribution free approach to calculate expected shortages during stock out situation. For future research, the model can be extended to imperfect production system with inspection errors and quality improvement. Another extension is possible by considering different shipping modes for lead time reduction in an imperfect production

environment. One can consider service level and budget constraint, multi-item and uncertain demand.

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