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RESEARCH ARTICLE

Analytical Solution of Linear Fractionally Damped Oscillator by Elzaki Transformed Method

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ABSTRACT

In this manuscript, linearly and fractionally damped oscillator equation is solved with the help of Caputo fractional derivatives. For $0 \le \alpha \le 1$ oscillation equations change from undamped to damped, whereas α represents the order of Caputo derivative. Analytical solution of nine different cases of critically damped, over-damped and undamped differential equations is found by Elzaki transformation. The oscillatory frequency of three cases of differential equation rises with the increase in damping order before falling to its limiting value provided by the ordinary oscillator damped equation whereas the frequency of oscillation of remaining six cases declines with the increase in derivative order (damping order).

Keywords: Elzaki transformation, Linear oscillator, Fractional damping, Oscillatory frequency and Caputo fractional derivative.

1. INTRODUCTION

The basic idea of this manuscript is to study the linear fractional damped oscillator [1], in which the order is restricted to $0 \le \alpha \le 1$, which is given in (1).

$$D_t^2 u + \delta_0 D_t^{\alpha} u + \omega^2 u = 0 \tag{1}$$

[2, 3] have studied the oscillation equation using generalized Mittag leffler functions. [4] The chaotic behavior of non-linear fractional oscillators is studied numerically. [5] Vigilant analytic study explains that the linear fractional equation with damping assists to determine the characteristics of the non-linear equation and this can be used for unswerving applications of fractional oscillations with damping. In this article, caputo fractional derivative formulation is castoff. For physical motives, caputo derivative is selected above the Riemann-Liouville derivative. The Elzaki transformations of two originations of fractional derivative for $0 < \alpha < 1$ are considered as in (2) and (3).

$$E[{}_0^{RL}D_t^{\alpha} f(t)] = \frac{1}{v^{\alpha}} T(v) - {}_0^{RL}D_t^{\alpha-1} f(t)|_0 \tag{2}$$

$$E[{}_0^{C}D_t^{\alpha} f(t)] = \frac{1}{v^{\alpha}} T(v) - f(t)|_0 \tag{3}$$

The continuous term that present in the Elzaki transform of caputo derivative is an initial assessment and it does not consists of any physical explanation. But this case does not satisfy for Riemann-Liouville derivative. Thus, it is said that, caputo fractional derivative seems more beneficial to demonstrate physical systems rather than Riemann-Liouville derivative. If the index of the fractional damping term becomes 3/2, then the equation is denoted as Bagley-Torvik equations, which displays the behavior of damped oscillation. Initially, this equation is derived to investigate the rigid plate motion in Newtonian fluid [6, 7]. The analytic explanation related to fractionally damped equation is established using Elzaki transform. In order to determine precise results and

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overcome certain problems of fractional equations, laplace transform method has been applied to the non-fractional equation. Nine individual cases are observed for fractionally damped equation whereas in contrast to this, only three cases are considered for non-fractional equation. In six cases of fractionally damped equation, damping effect rises with increase in exponent of fractional derivative. However, in other three cases, as the fractional derivative index rises, the damping frequency also rises until a peak value is attained, after which it starts decreasing to its non-fractional limit. The reason for the rise in oscillation frequency is not stated clearly.

2. NON-FRACTIONAL CASE

The basic idea is to determine the solution of linear damped oscillator, which is analyzed using Elzaki transform technique given in (4).

$$D_t^2 u + \delta D_t u + \omega^2 u = 0 \tag{4}$$

where the constants δ and ω are considered as real and positive. δ and ω^2 is the damping and reinstating force expressed in per unit mass respectively. In either cases (both fractional and non-fractional), the succeeding initial approximation is used as given in (5) and (6).

$$u(0) = u_0 \tag{5}$$

$$D_t u(0) = u_1 \tag{6}$$

Transform (4) together with the initial conditions (4) and (5) gives (7), where T(v) is given in (8).

$$\frac{1}{v^2}T(v) - T(0) - vT(0) + \delta\left(\frac{1}{v}T(v) - vT(0)\right) + \omega^2 T(v) = 0 \tag{7}$$

$$T(v) = \frac{u_0 v^2}{v^2\omega^2 + \delta v + 1} + \frac{\delta u_0 v^3 + u_1 v^3}{v^2\omega^2 + \delta v + 1} \tag{8}$$

(7) can be overturned using tables; however, to explain the problem that occurs later, (7) is overturned using the complex inversion integral. Whole numbers is the order of variable in both terms, hence counter integral does not have a branch cut and bromwich contour is used, which is defined as in (9).

$$u(t) = Residue - \frac{1}{2\pi i}\int e^{\frac{-t}{v}} T(v)dt \tag{9}$$

Bromwich contour commences at $\gamma - i\infty$ and then moves up perpendicularly to $\gamma + i\infty$ (whereas γ is selected in such a way that all poles lies on the left of perpendicular contour line, due to this all the poles are confined within a contour) and then trips in a semi circle (counter clockwise on the left side) and comes back to $\gamma - i\infty$. Contour integral has no involvement regarding this problem. Residue contributes the only input, which is obtained from the roots of quadratic (10).

$$v^2\omega^2 + \delta v + 1 = 0 \tag{10}$$

Three different cases of quadratic equation are,

1. $\delta > 2\omega$; 2 real roots are unequal and negative and is shown in (11).

$$v_{1,\,2} = \frac{-\delta \pm \sqrt{\delta^2 - 4\omega^2}}{2\omega^2} \tag{11}$$

2. $\delta = 2\omega$; 2 real roots are repeated and negative which is given in (12).

$$v_3 = -\frac{\delta}{2\omega^2} \tag{12}$$

3. $\delta = 2\omega$; 2 complex roots with negative real parts is shown in (13).

$$v_{4,\,5} = \frac{-\delta \pm i\sqrt{4\omega^2 - \delta^2}}{2\omega^2} \tag{13}$$

Note that the index is one for case (1) and (3) respectively whereas for case (2), the index is two. In addition to this, the poles lie on imaginary axis at $\pm i\omega$, if there is no damping as shown in figure 1.

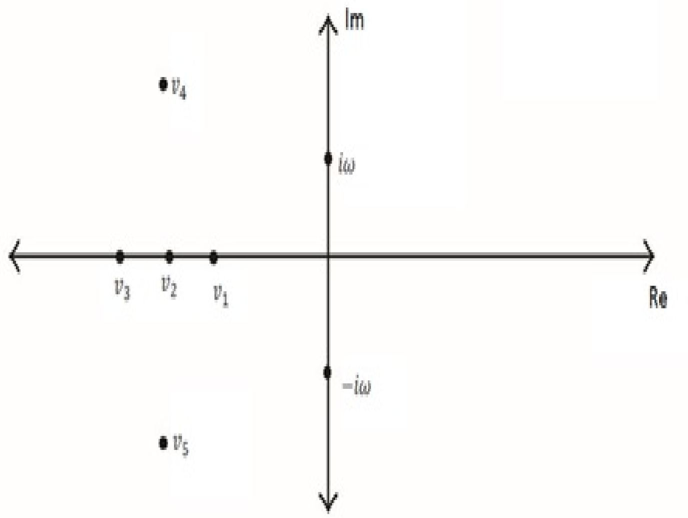

Figure 1. Real and imaginary plots

For case (1) the residue and the u(t) is given in (14) and (15) repectively.

$$Residue = \lim_{v \to v_{1,\,2}} (v - v_{1,\,2}) e^{\frac{t}{v}} \left(\frac{u_0 v^2}{v^2 \omega^2 + \delta v + 1} + \frac{\delta u_0 v^3 + u_1 v^3}{v^2 \omega^2 + \delta v + 1} \right) \tag{14}$$

$$u(t) = \frac{e^{\frac{t}{v_1}}}{2v_1 + \delta} (v_1 u_0 + u_1 + \delta u_0) + \frac{e^{\frac{t}{v_2}}}{2v_2 + \delta} (v_2 u_0 + u_1 + \delta u_0) \tag{15}$$

Since v_1 and v_2 are negative, residue declines exponentially, hence referred as over-damped case. In a similar way, residue for case (3) is also calculated. Since the poles are complex, the exponential function is represented based on sine and cosine components with exponential damping factor as in (16).

$$u(t) = e^{-\frac{\delta t}{2}} \left(u_0 \cos\left(\sqrt{\omega^2 - \frac{\delta^2}{4}} t \right) + \frac{2u_1 + \delta u_0}{2\sqrt{\omega^2 - \frac{\delta^2}{4}}} \sin\left(\sqrt{\omega^2 - \frac{\delta^2}{4}} t\right) \right) \tag{16}$$

where $\rho = \sqrt{\omega^2 - \frac{\delta^2}{2}}$ and $\alpha = \frac{\delta}{2}$. It is noted that effective angular frequency ρ, which is smaller than undamped angular frequency results in existence of damping, which also slows down the oscillations. By assessment, for both cases, we might observe that they might have infinite period or zero frequency. The index pole is 2 for the second case, and (17) gives the residue,

$$\lim_{v \to -\omega} \frac{d}{dv} \left((v + \omega)^2 e^{\frac{t}{v}} \left(\frac{v u_0 + u_1 + \delta u_0}{v^2 \omega^2 + \delta v + 1} \right) \right) \tag{17}$$

Remember that $\delta = 2\omega$ permits the denominator to be factorized and the limits are given in (18) and (19).

$$\lim_{v \to -\omega} \frac{d}{dv} (e^{\frac{t}{v}} (v u_0 + u_1 + 2\omega u_0)) \tag{18}$$

$$u(t) = (e^{\omega t})(u_0 + (u_1 + \omega u_0)t) \tag{19}$$

Generally, it is named as critically damped case.

3. FRACTIONAL CASE

In this case, as considered in [1], δ has exponent v - 2. So, the total exponent of second term remains unchanged as given in [4]. Two cases based on $0 < \alpha < 1$ and $0 < \alpha < 2$ are considered. Applying Elzaki transform on [1], we get (20).

$$T(v) = \frac{u_0 v^{\alpha}}{v^{\alpha}\omega^2 + v^{\alpha-2} + \delta} + \frac{\delta u_0 v^{2\alpha-1} + u_1 v^{\alpha+1}}{v^{\alpha}\omega^2 + v^{\alpha-2} + \delta} \tag{20}$$

A branch cut is required on negative real axis due to the fractional exponents on the complex variable s if the above (20) is overturned through contour integral. As a result, Hankel contour is used, which begins at $\gamma - i\infty$ and moves up perpendicularly to $\gamma + i\infty$ (whereas γ is again selected in such a way that all poles lie on left of the perpendicular curve line) then quarter circular arc in trips (to the left) lies just on top of real negative axis (i.e., $-\infty$). A cut on the curve that goes into the origin (followed by real negative axis) goes clockwise about origin (lies underneath the real negative axis) and then it moves back out to $-\infty$. This contour has additionally some part of the circular arc to $\gamma - i\infty$.

$$v^{\alpha}\omega^2 + v^{\alpha-2} + \delta = 0 \tag{21}$$

Trivial problem does not exist for any random value of α. To accomplish this, several solutions are determined. Let $v = e^{1\theta}$, then (21) is expressed as real and imaginary part, which is given as in (22) and (23).

$$\omega^2 r^{\alpha}\cos(\alpha\theta) + r^{\alpha-2}\cos((\alpha-2)\theta) + \delta = 0 \tag{22}$$

$$\omega^2 r^{\alpha}\sin(\alpha\theta) + r^{\alpha-2}\sin((\alpha-2)\theta) = 0 \tag{23}$$

(22) (positive real axis) is not satisfied when $\theta = 0$ due to the addition of nonzero positive terms. Similarly, (23) is not satisfied (negative real axis) when $\theta = \pi$. Thus it is shown that no result exists on positive or negative imaginary axes with identical arguments $0 < v < 1$. It is also shown that solutions do not lie in right half plane (i.e., both terms are always positive in the second equation). But in contrast to this, if solution exists in right half plane, then they are supposed to be ordered pairs, complex conjugates in $\frac{\pi}{2} < \theta < \pi$ and $\frac{-\pi}{2} < \theta < -\pi$. To attain a resolution, (23) is solved to determine r and this value is included in the first equation (positive values of θ are considered first).

$$\omega^2 r^{\alpha}\left(-\delta\frac{(\sin(\alpha\theta))^{\alpha}}{(\sin(2\theta))^2}\right)^{\frac{2}{(2-\alpha)}}\cos(2\theta) + \delta\left(-\delta\frac{(\sin(\alpha\theta))^{\alpha}}{(\sin(2\theta))^2}\right)^{\frac{\alpha}{(2-\alpha)}}\cos(\alpha\theta) + \omega^2 = 0 \tag{24}$$

For the constrained angular range $\frac{\pi}{2} < \theta < \pi$, sin $(2\theta$) in (24) is negative. Hence the root argument becomes positive. θ cannot be determined if values of α, δ and ω are provided. (24) is further simplified as (25).

$$\left(\frac{(\sin(\alpha\theta))^{\alpha}}{(\sin(2\theta))^2}\right)^{\frac{1}{(2-\alpha)}}\sin((2-\alpha)\theta) = \left(\frac{\omega}{(\delta)^{\frac{1}{(2-\alpha)}}}\right)^2 \tag{25}$$

$\sin((2-\alpha)\theta)$ must be positive for the true solution of (25). This is possible only if the restricted domain $\frac{\pi}{2} < \theta < \frac{\pi}{2-\alpha}$. θ doesn't depend on the values of ω, δ and α. It is mentioned that a value of θ can be determined all the time satisfying (25). The two limits as in (26) and (27) are considered.

$$\lim_{\theta\to\frac{\pi}{2}^+}\left(\frac{(\sin(\alpha\theta))^{\alpha}}{(\sin(2\theta))^2}\right)^{\frac{1}{(2-\alpha)}}\sin((2-\alpha)\theta) = \infty \tag{26}$$

$$\lim_{\theta\to\frac{\pi}{2-\alpha}^-}\left(\frac{(\sin(\alpha\theta))^{\alpha}}{(\sin(2\theta))^2}\right)^{\frac{1}{(2-\alpha)}}\sin((2-\alpha)\theta) = 0 \tag{27}$$

As the LHS of (25) is continuous in θ and we have two limits above (27), it is guaranteed that there will be at least one solution to (25) and for residue control, there will be only two poles. We demonstrate that θ declines monotonically over the constrained field in the LHS of (25). This shows that only one solution is possible for (25), and for residue controls, there are two poles. To demonstrate the monotonic decline in value of θ in LHS of (25), the derivative of LHS of (25) with respect to θ is required to be exhibited as negative, as in (28).

$$\frac{\partial}{\partial\theta}\left\{\left(\frac{(\sin(\alpha\theta))^{\alpha}}{(\sin(2\theta))^{2}}\right)^{\frac{1}{(2-\alpha)}}\sin((2-\alpha)\theta)\right\}<0 \tag{28}$$

By neglecting positive feature, derivatives are calculated and is given by (29).

$$\alpha^2\sin^2(2\theta)-4\alpha\sin(2\theta)\sin(\alpha\theta)\cos(\alpha\theta)cos((2-\alpha)\theta)+4\sin^2(\alpha\theta)>0 \tag{29}$$

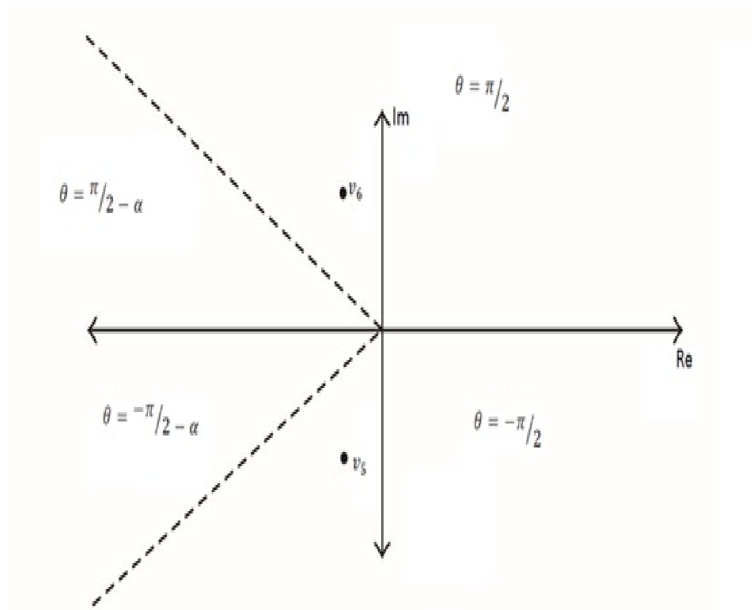

Figure 2.Position of the roots

Constrained field $\sin(\alpha\theta)>0,\ \sin(2\theta)<0$ and $\cos((2-\alpha)\theta)\leq 1$. This decrease to (29)

$$\alpha^2\sin^2(2\theta)-4\alpha\sin(2\theta)\sin(\alpha\theta)+4\sin^2(\alpha\theta)>0 \tag{30}$$

(30) is now transformed into a perfect square as in (31) to verify the proclamation made in (28).

$$(\alpha\sin(2\theta)-2\sin(\alpha\theta))^2>0 \tag{31}$$

Hence, the left side of (25) declines monotonically with upper bound ∞ and lower bound θ. In order to reaffirm this, there is only one solution having a positive angle for (25) and value of θ must lie in the interval $\theta=-\frac{\pi}{2}<\theta<\frac{\pi}{2-\alpha}$. Accordingly, for residue calculations, there exists two poles, which are complex conjugates to each other. Recurrent root does not exist for fractionally damped equation, whereas it is possible when the derivative order is one. Figure 2 illustrates the position of the roots graphically. The two poles are represented as in (32).

$$v_{6,\,7}=\beta\pm i\sigma=re^{i\theta} \tag{32}$$

where β and $i\sigma$ are resolute values of r and θ that gratify (25) in the typical way, $r=\sqrt{\beta^2+\sigma^2}$ and $\tan\theta=\frac{\sigma}{\beta}$. If β is negative, v_7 is the complex conjugate of v_6, then these two results lie in the second and third quadrants respectively. Also it is noted that, β plays the role of $\frac{\delta}{2}$ from the non-fractional case. The residue with the index pole one is given as in (33).

$$Residue = \lim_{v\to v_6}(v-v_6)e^{\frac{t}{v}}\left(\frac{vu_0+u_1+\delta u_0 v^{\alpha-1}}{v^2+\delta v^{\alpha}+\omega^2}\right)+\lim_{v\to v_7}(v-v_7)e^{\frac{t}{v}}\left(\frac{vu_0+u_1+\delta u_0 v^{\alpha-1}}{v^2+\delta v^{\alpha}+\omega^2}\right)$$

$$e^{\frac{t}{v_6}}\left(\frac{v_6u_0+u_1+\delta u_0 v_6^{\alpha-1}}{2v_6+\alpha\delta v_6^{\alpha-1}}\right)+e^{\frac{t}{\overline{v_6}}}\left(\frac{\overline{v_6}u_0+u_1+\delta u_0\overline{v_6}^{\alpha-1}}{2\overline{v_6}+\alpha\delta\overline{v_6}^{\alpha-1}}\right) \tag{33}$$

where v_7 is replaced by $\overline{V_6}$ in (34).

$$2e^{\beta t}\cos(\alpha t)\left(\frac{u_0(2r^2+\alpha\delta r^{2\alpha-2}+\delta r^{\alpha}(\alpha+2)\cos((\alpha-2)\theta)+u_1(2r\cos(\theta))+\alpha\delta r^{\alpha-1}cos((\alpha-1)\theta))}{4r^2+4\alpha\delta^{\alpha}r^{\alpha}\cos((2-\alpha)\theta)\alpha^2\delta^{2\alpha}r^{2\alpha-2}}\right)$$

$$+2e^{\beta t}sin(\alpha t)\left(\frac{u_0(\delta r^{\alpha}(\alpha-2)\sin((\alpha-2)\theta))+u_1(2r\sin(\theta)+\alpha\delta r^{\alpha-1}\sin((\alpha-1)\theta)}{4r^2+4\alpha\delta^{\alpha}r^{\alpha}\cos((2-\alpha)\theta)+\alpha^2\delta^{2\alpha}r^{2\alpha-2}}\right) \tag{34}$$

(35) shows the trails including the negative real axis for the contour integral.

$$\frac{\delta}{\pi}\int_0^{\infty}\frac{(Ru_0-u_1)\sin(\alpha\pi)+\frac{u_0}{R}(R^2+\omega^2)\sin(\alpha\pi-\pi)}{(R^2+\omega^2)^2+2\delta R^{\alpha}(R^2+\omega^2)\cos(\alpha\pi)+(\delta R^{\alpha})^2}e^{-Rt}R^{\alpha}dR \tag{35}$$

The solution to (1) deducted from (32) is given in (33). This might look difficult but it explains the generalized arrangement.

$$u(t)=Ae^{\beta t}\cos(\sigma t)+Ae^{\beta t}\cos(\sigma t)-$$
$$\frac{\delta}{\pi}\int_0^{\infty}\frac{(Ru_0-u_1)\sin(\alpha\pi)+\frac{u_0}{R}(R^2+\omega^2)\sin(\alpha\pi-\pi)}{(R^2+\omega^2)^2+2\delta R^{\alpha}(R^2+\omega^2)\cos(\alpha\pi)+(\delta R^{\alpha})^2}e^{-Rt}R^{\alpha}dR \tag{36}$$

Notice that the decay function (35), goes to zero if v in (1) drives to zero or one. This damping factor $e^{\beta t}$ as in (36) is identical to the non-fractional case $e^{\frac{-\delta t}{2}}$. Meanwhile, the poles have non-zero imaginary and real parts, and for residue calculation, it is not possible to have three dissimilar cases as observed in non-fractional case (under-damped, critically damped and over-damped).

4. OSCILLATION FREQUENCY

The oscillatory frequency is the element of solution $\sigma = Im(v_6)$. As the frequency changes, the fractional damping index also changes and the reason for this modification is explained below. When v tends to zero, the undamped oscillator based on frequency is given as in (37).

$$\sigma=\sqrt{\delta+\omega^2} \tag{37}$$

Three different cases such as overdamped, critically damped and underdamped are analyzed when α tends to one as given in section 2. As a result, the frequency might be zero or non-zero as in (38) and (39).

$$\sigma=0,\delta\geq 2\omega \tag{38}$$

$$\sigma=\frac{\sqrt{4\omega^2-\delta^2}}{2},\delta<2\omega \tag{39}$$

For $0<\alpha<1$, frequency will always be non-zero. Note that $0<\frac{\sqrt{4\omega^2-\delta^2}}{2}<\sqrt{\delta+\omega^2}$. In the non-fractional case, cumulative δ results in monotonic decrease of oscillator frequency till the critical cases (over-damped and critically damped) are touched and period of oscillation becomes infinite, whereas in fractional case, oscillator frequency $\sigma = Im(v_6)$ depends on the derivative order α, δ and ω. In order to show the dependency of σ on these three factors, we assume s as a function of α on $0 \leq \alpha \leq 1$, and is defined as in (40).

$$v^{\alpha}\omega^2 + v^{\alpha-2} + \delta \tag{40}$$

For a fixed value of δ and ω (both are positive), considering upper half plane for v, v has one to one correspondence on α on $0 \leq \alpha \leq 13$. A branch cut on the negative real axis exists as a result of fractional index triggering, hence v does not follow one to one at $\alpha = 1$. The derivative of (40) with reference to α given as $\frac{dv}{d\alpha}$ (recall, δ and ω are being kept fixed) is considered as in (41).

$$\frac{dv}{d\alpha} = \frac{\delta v^{\alpha+1}\ln(v)}{2v^2 + \delta\alpha v^{\alpha}} \tag{41}$$

Imaginary part of above equation is given in (42).

$$\frac{d\sigma}{d\alpha} = Im\frac{dv}{d\alpha} \tag{42}$$

Precisely, the equation at $\alpha = 0$ is considered as in (43).

$$\frac{d\sigma}{d\alpha}|_{v=0} = Im\frac{v(v^2+\omega^2)\ln(v)}{2v^2 - v(v^2+\omega^2)}|_{v=0} = \delta\frac{\ln(\delta+\omega^2)}{4\sqrt{\delta+\omega^2}} \tag{43}$$

when

$\delta + \omega^2 > 1$ The frequency rises with rising damping order.

$\delta + \omega^2 = 1$ The frequency does not vary with rising damping order.

$\delta + \omega^2 < 1$ The frequency declines with rising damping order.

The obtained answer is not completely as predicted. The oscillation freguency rises before declining in the first case. There are a number of values of α for which the fractional damping causes the oscillations to go quicker than oscillations without damping (still damping is the cause of decline in amplitude). Individually, three cases discussed above can become three cases with non-fractional damping by letting $\alpha \to 1$ ((10)-(12)). Therefore, there are nine different cases for the linear fractional oscillator with damping. Some graphical representation results of imaginary part in (40) (the frequency of oscillation) with respect to α, δ and ω are obtained. The frequency of oscillation and the derivative order lie on the vertical and horizontal axis respectively in all the three cases as shown in figures 3, 4, 5 and it also resembles the three cases (under damped, critically damped, over damped) of damped oscillator based on derivative order of whole numbers.

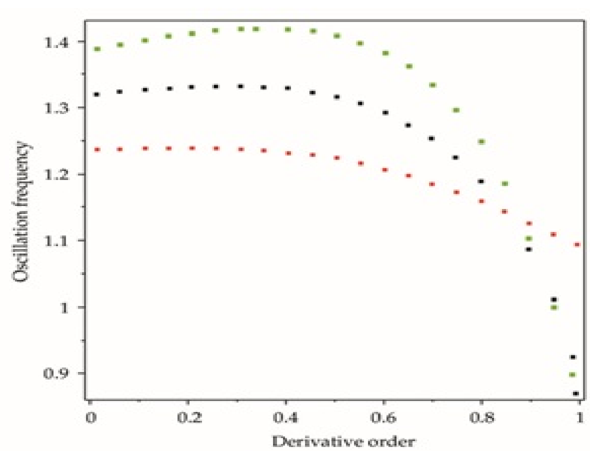

Figure 3.Illustrative figure for case I, $\delta + \omega^2 > 1$. Green curve is for $\delta = \omega = 1$ black curve is for $\delta = 2$ and $\omega = 1$, red curve is for $\delta = 3$ and $\omega = 1$

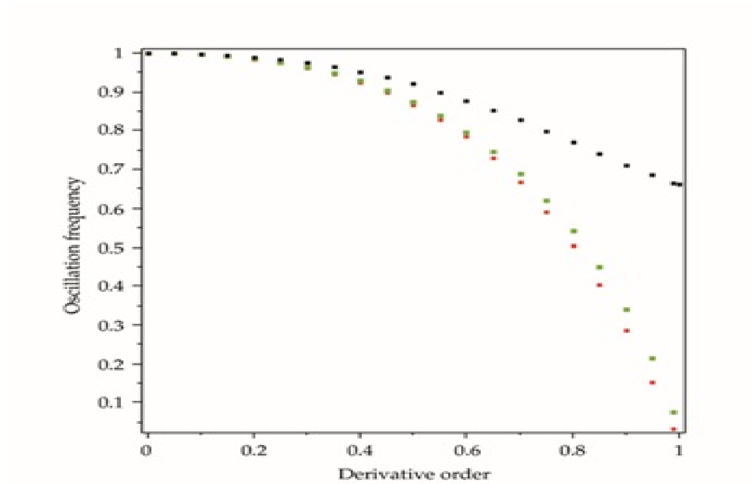

Figure 4.Illustrative figure for case II, $\delta+\omega^2>1$, a line is flat at the beginning. Red curve is for $\delta=2(2-1)$ and $\omega=\frac{1}{2}$, green curve is for $\delta=\frac{1}{2}$ and $\omega=\frac{1}{2}$, black curve is for $\delta=\frac{15}{16}$ and $\omega=\frac{1}{4}$

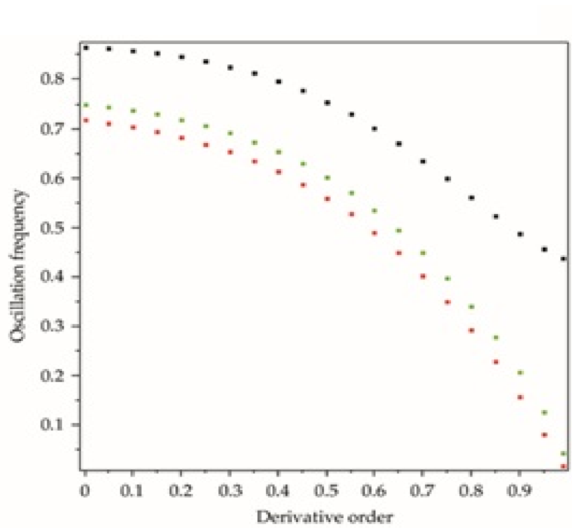

Figure 5.Illustrative graph for case III, which declines at the beginning. The green curve is for $\delta=\frac{1}{2}$ and $\omega=\frac{1}{8}$, the black curve is for $\delta=\frac{1}{2}$ and $\omega=\frac{1}{4}$, and the red curve is for $\delta=\frac{1}{2}$, $\omega=\frac{1}{2}$

5. CONCLUSION

Thus, the Elzaki transform method obtains the analytical solution of linear fractionally damped oscillator. Solution obtained by proposed method is identical to non-fractional decomposed oscillations along with the inclusion of supplementary function. Nine discrete cases are found in contrast to typical three cases for ordinary damped oscillator. Increasing the damping order, damping term rises until it attains the highest value and then decreases. Physical aspects of the rise in oscillation are still needed to be determined.

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