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RESEARCH ARTICLE

An Inventory Model with Backorder Price Discount and Stochastic Lead Time

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ABSTRACT

This study investigates an inventory model to reduce the manufacturing setup cost, where order quantity, setup cost, reorder point, backorder price-discount rate, and lead time are decision variables. The model assumes a lead time dependent backorder rate, where the lead time demand is stochastic in nature, which follows a normal distribution. There is a proposition to obtain the optimal solutions for the model. Some numerical examples and sensitivity analysis are presented to illustrate the proposed model. The numerical results are compared with the existing results from literature, and the results indicate significant savings over the existing total cost.

Keywords: Setup cost reduction, Backorder price-discount, Stochastic lead time, Distribution free approach.

1. INTRODUCTION

Traditionally, basic inventory models assume that setup cost is fixed or constant. Through the Japanese experience of Just-In-Time (JIT) production, one can learn that, in many manufacturing systems including job shops, batch shops, and flow shops, the setup cost can be reduced by investing capital, i.e, the setup cost is assumed to be variable rather than constant. A stock-out situation occurs whenever sufficient stock does not exist to fulfill the replenishment order. Some customers may wait for the item, while others may not instead purchasing the item from another source. This indicates that the supplier has lost the opportunity to earn more profit, disappointed customers, and probably put some doubts in customer's mind about the nature of the storage capacity of the supplier. On the other hand, backordering could result in handling and expediting costs to reduce the lead time. In order to compensate customers for the inconvenience of waiting, the idleness of equipment, or even lost production during the stock out period, the supplier may offer a variable price discount on the stock out items depending on the seriousness of the backorder condition. Thus, both the backorder price-discount and lead time appear to be negotiable in such a way that the supplier may reduce the present and future losses, and the customers may be able to obtain items as soon as possible to resume production. This model investigates the joint effect of setup cost reduction and backorder price-discount in which order, quantity, reorder point, lead time, setup cost and backorder price-discount are decision variables.

In this competitive environment, one of the ways to attract customers is the use of shorter lead time and quick service. Lead time reduction can be defined as the process of decreasing lead time at an increased cost. If an industry can deliver products quickly compared to others, then it may increase the likelihood of receiving future orders. Hence, lead time reduction is an important factor in every industry sector. Reduced lead time can decrease the stock out loss and improve the customer's satisfaction level. There are several inventory models that consider constant lead time. But in reality, JIT cannot be implemented in all situations, thus, researchers should consider variable lead time.

[1] considered an inventory model to analyze interactive decisions on lead time between a manufacturer and a retailer. [2] considered a continuous review inventory model with shortages where the quantity received is uncertain. In this, the lead time crashing cost is an exponential function of lead time and order processing cost, and lost sales rate is logarithmic function of capital investment.

It is to be noted that the above existing literature focused on lead time reduction in which the setup cost was

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treated as fixed or constant. Japanese experience of Just-In-Time (JIT) production said that in many manufacturing systems including job shops, batch shops, and flow shops, setup cost can be reduced by investing capital. That is, the setup cost is assumed to be variable rather than constant. By reducing the setup cost, one can reduce the total inventory cost or can gain more profit. Regarding cost reduction policy, [3] developed an investment concept to reduce setup cost in an inventory model. [4] developed a continuous review inventory model with lead time and ordering cost reduction. [5] derived a note on periodic review inventory model with a controllable setup cost and lead time. [6] discussed a continuous review inventory model to show the curve-effect on setup cost reduction involving the controllable lead time, mixture of backorder, and partial lost sales. [7] developed an inventory model to investigate the impact of setup cost reduction involving controllable backorder rate and variable lead time with a service level constraint. [8] considered an integrated vendor-buyer supply chain model with vendor's setup cost reduction.

Table 1.Comparison of various contributions

References	Variable lead time	Order quantity	Setup cost reduction	Backorder price discount	Variable safety factor
[2]	✓	✓		✓	
[3]		✓	✓		
[4]	✓	✓			
[5]	✓		✓		
[6]	✓	✓	✓		✓
[7]	✓	✓	✓		
[8]	✓	✓	✓		✓
[9]	✓	✓		✓	
[10]	✓	✓			
[11]	✓	✓		✓	
[12]	✓	✓		✓	
[13]	✓	✓		✓	
[15]	✓	✓		✓	
[16]	✓	✓		✓	✓
[18]	✓	✓	✓	✓	
[19]				✓	
[20]				✓	
[22]	✓				
[24]	✓	✓		✓	✓
This paper	✓	✓	✓	✓	✓

In the basic inventory model, it is often assumed that shortages are either completely backlogged or completely lost. But in real life situation, some customers may favor for backorders for their desired items, while others may refuse to take backorders. During shortage, there are many factors that can increase customers willingness of accepting backorders. One of those is an offering of a backorder price-discount on shortages items to customers. Providing of more backorder prices discount on stock out items can make the customers more willing to wait for the desired items. By providing backorder price-discount on stock out items, the supplier can sold more items that can reduce the holding cost. In this direction, [9] presented an EOQ model with a backorder price-discount and a variable lead time. [10] developed an inventory model with backorders and lost sales for variable lead time demand. They considered the total amount of stock out as a mixture of backorders and lost sales during the stock out period.

[11] investigated a continuous review inventory model with a mixture of backorders and lost sales in which the order quantity, backorder discount and lead time are considered as decision variables. [12] developed an integrated inventory model with a controllable lead time and backorder discount. [13] discussed an inventory model with a controllable negative exponential backorder rate and a service level constraint. [14] presented an integrated production-inventory model for ameliorating and deteriorating items with partial backordering and time discounting. [15] developed an optimal inventory policy involving backorder price-discounts and variable lead time demand. [16] presented a distribution free model with backorder price-discount in which the lead time and ordering cost reductions are inter-related. [17] investigated a continuous review model with backorder price-discount and variable lead time to effectively increase investment and to reduce the joint expected annual total cost. [18] developed a periodic review inventory model with backorder price-discounts, where shortages are partially backlogged. [19] explained a deterministic inventory model with permissible delay-in-payments and price-discount on back-

orders. [20] analyzed an inventory model with delays in payment and price-discount offers in the supply chain system, and provided a concrete solution procedure to locate the optimal solution. [21] used normal distribution to determine probabilistic rate of imperfect production along with other distribution. [22] considered a periodic review inventory model with controllable lead time and partially backlogging, in which the backorder rate is dependent on the backorder price-discount and the length of the protection interval. We summarize our contribution compared with other models in table 1.

The present study investigates a continuous review inventory model with order quantity, reorder point, backorder price discount, setup cost, and lead time as decision variables. The goal of this research is to minimize the total related cost by optimizing the order quantity, setup cost, backorder price discount, reorder point, and lead time, simultaneously. Furthermore, one efficient computational algorithm is implemented. The rest of this paper is organized as follows: assumptions and notation are stated in section 2. In section 3, the mathematical model is developed. Numerical examples and sensitivity analysis are provided in section 4. Finally, conclusions are presented in section 5.

2. NOTATION AND ASSUMPTIONS

We use the following notation to develop the model.

Decision variables

Q order quantity (units)
A setup cost per setup, after setup cost reduction ($/setup)
L length of the lead time (days)
π_x backorder price-discount per unit offered by supplier ($/unit)
k safety factor

Parameters

D annual average demand (units/year)
A_0 initial setup cost per setup, before setup cost reduction ($/setup)
h holding cost per unit per year ($/unit/year)
π_0 marginal profit per unit ($/unit)
α annual fractional cost of the capital investment ($/year)
C(L) lead time crashing cost function
β backorder ratio
r reorder point
β_0 upper bound of the backorder ratio
σ standard deviation of the lead time demand
u_i i th component of lead time with u_i as minimum duration (days), i=1, 2,...., n
v_i i th component of lead time with v_i as normal duration (days), i=1, 2,...., n
m_i i th component of lead time with m_i as crashing cost per day, i=1, 2,..., n
X lead time demand, which has a distribution function F with finite mean DL and standard deviation $\sigma\sqrt{L}$
E(x) mathematical expectation of x
x^+ max $\{x,0\}$
$E(X-r)^+$ expected shortage quantity at the end of the cycle.

We assume the following assumptions to develop our model.

1. Inventory is continuously monitored. The model considers a single type of item. The model does not consider production directly, but it involves the assembling of products such that the setup cost is involved. Replenishment occurs whenever the inventory level drops to the reorder point r.
2. The reorder point is determined by r=DL + k$\sigma\sqrt{L}$, where DL= expected demand during the lead time, and k$\sigma\sqrt{L}$= safety stock [23, 24].
3. It is possible to reduce the setup cost by using an initial investment function. Thus we assume a logarithmic investment function to reduce the setup cost. The investment function is convex for all A with the restriction. It can be easily shown that the characteristics of the model obtained using a logarithmic investment function remain essentially unaffected if we use any other investment function like sub-linear power investment function.
4. The lead time L consists of n mutually independent components. The ith component has a minimum duration u_i in days, normal duration v_i in days, and a crashing cost per day m_i. Further, we rearrange m_i as $m_1 \le m_2 \le m_3 \le \ldots \le m_n$. Then, it is clear that the reduction in lead time should first occur on component 1 (because it has the

minimum unit crashing cost), and then component 2, and so on.
5. We suppose $L_0 = \sum_{j=1}^{n} v_j$ and L_i as [13],

$$L_i = \sum_{j=1}^{n} v_j - \sum_{j=1}^{i} (v_j - u_j)$$

where i=1, 2, ..., n, and the lead time crashing cost function C(L) per cycle is given by,

$$C(L) = m_i(L_i - L) + \sum_{j=1}^{i-1} m_j(v_j - u_j)$$

6. The backorder ratio β is considered to be variable and is proportion to the price-discount offered by supplier per unit π_x. Thus $\beta = \beta_0\pi_x/\pi_0$, $0 \le \beta_0 \le 1$ and $0 \le \pi_x \le \pi_0$ [9].

3. MATHEMATICAL MODEL

The model is considered based on the probability distribution of lead time demand (normal distribution model).

3.1. Normal distribution model

This model assumes that inventory is continuosly monitored. That is, the buyer orders quantity Q, when the inventory level drops to the reorder point r. Hence, the reorder point is determined by r = DL + K$\sigma\sqrt{L}$, where DL = expected demand during the lead time, K$\sigma\sqrt{L}$ = safety stock. Thus, before receving an order, the inventory is r - DL and after receiving the order, the inventory is Q + r - DL. Hence, the holding cost is $(h\frac{Q}{2} + r - DL)$ As following assumption 5, the lead time crashing cost is $\frac{D}{Q}C(L)$ and the setup cost is $\frac{AD}{Q}$, where the demand, setup cost and lot size are deterministic.

This model considers that demand X during the lead time has a probability density function F(x) with mean DL and standard deviation $\sigma\sqrt{L}$. It is noted that r = DL + k$\sigma\sqrt{L}$ and that the expected shortage quantity at the end of a cycle is given by,

$$E(X-r)^+ = \int_r^\infty (x-r)dF(x) = \sigma\sqrt{L}\psi(k)$$

where $\psi(k) = \phi(k) - k[1-\Phi(k)]$ and ϕ and Φ denote the standard normal probability density function (pdf) and distribution function (df) respectively. The Total Expected Aannual Cost (TEAC) is given in (1).

$$\begin{aligned} TEAC(Q,r,\beta,L) &= Setup\ cost + holding\ cost + stock-out\ cost + lead\ time\ crashing\ cost \\ &= \frac{AD}{Q} + h\left[\frac{Q}{2} + r - DL + (1-\beta)E(X-r)^+\right] + \frac{D}{Q}[\pi_x\beta + \pi_0(1-\beta)]E(X-r)^+ + \frac{D}{Q}C(L) \end{aligned} \tag{1}$$

Submitting r = DL + k$\sigma\sqrt{L}$ and $\beta = \beta_0\pi_x/\pi_0$ in (1), we obtain (2).

$$\begin{aligned} TEAC(Q,r,\pi_x,L) = \frac{AD}{Q} &+ h\left[\frac{Q}{2} + k\sigma\sqrt{L} + \left(1 - \frac{\beta_0\pi_x}{\pi_0}\right)\sigma\sqrt{L}\psi(k)\right] \\ &+ \frac{D}{Q}\left[\frac{\beta_0{\pi_x}^2}{\pi_0} + \pi_0 - \beta_0\pi_x\right]\sigma\sqrt{L}\psi(k) + \frac{D}{Q}C(L) \end{aligned} \tag{2}$$

Thus, the safety factor k and the backorder price discount π_x offered by the supplier per unit can be treated as decision variables.

3.2. Investment in setup cost reduction

Now the effect of investment on setup cost reduction is investigated. Additional investment is a good strategy to reduce setup cost. As for example, an investment of $200 can reduce the original setup cost of $100 by ten percent to $90, an investment of the same amount can reduce the setup cost of $90 to $81, and so on. Therefore, the best way to reduce the setup cost is by using some initial investments. According [3], a logarithmic investment function $I_A(A)$ is assumed to reduce the setup cost, and it can be expressed as,

$$I_A(A) = B\ln\left(\frac{A_0}{A}\right), 0 < A \leq A_0$$

where $B = \frac{1}{\delta}$and δ= percentage decrease in A per dollar increase in $I_A(A)$. From the above function, it is noted that the setup cost level A$\in$ (0, A_0]. This implies that, if the optimal setup cost obtained does not satisfy the restriction on A, then no setup cost reduction investment is made. For this special case, the optimal setup cost is the original setup cost.

Hence, the expected annual total cost can be expressed as in (3).

$$TEAC(A,Q,k,\pi_x,L) = \alpha I_A(A) + TEAC(Q,k,\pi_x,L) = \alpha B\ln\left(\frac{A_0}{A}\right) + \frac{AD}{Q}$$
$$+h\left[\frac{Q}{2} + k\sigma\sqrt{L} + \left(1 - \frac{\beta_0\pi_x}{\pi_0}\right)\sigma\sqrt{L}\psi(k)\right] + \frac{D}{Q}\left[\frac{\beta_0\pi_x}{{\pi_0}^2} + \pi_0 - \beta_0\pi_x\right]\sigma\sqrt{L}\psi(k) + \frac{D}{Q}C(L) \tag{3}$$

Therefore, the aim is to minimize TEAC with respect to five decision variables and two different constraints as in (4).

$$MinTEAC^N(A,Q,k,\pi_x,L) = \alpha B\ln\left(\frac{A_0}{A}\right) + \frac{AD}{Q}$$
$$+h\left[\frac{Q}{2} + k\sigma\sqrt{L} + \left(1 - \frac{\beta_0\pi_x}{\pi_0}\right)\sigma\sqrt{L}\psi(k)\right] + \frac{D}{Q}\left[\frac{\beta_0\pi_x}{{\pi_0}^2} + \pi_0 - \beta_0\pi_x\right]\sigma\sqrt{L}\psi(k) + \frac{D}{Q}C(L) \tag{4}$$

subject to

$$0 < A \leq A_0$$
$$0 \leq \pi_x \leq \pi_0$$

where $\overline{\pi} = \pi_0 - \beta_0\pi_x + \beta_0\pi_x^2/\pi_0$ and $TEAC^N(A,Q,k,\pi_x,L)$ is the expected annual total cost when demand during lead time is normally distributed. It is a non-linear program and in order to solve this type of problem, the constraints $0 < A \leq A_0$ and $0 \leq \pi_x \leq \pi_0$ are relaxed. Now, calculating the first order partial derivatives of $TEAC^N(A,Q,k,\pi_x,L)$ with respect to A, Q, k, π_x, and L, respectively, one can obtain as in (5)-(9).

$$\frac{TEAC^N}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{D\overline{\pi}\sigma\sqrt{L}\psi(k)}{Q^2} - \frac{D}{Q^2}C(L) \tag{5}$$

$$\frac{TEAC^N}{\partial k} = h\sigma\sqrt{L} - \left[h\left(1 - \frac{\beta_0\pi_x}{\pi_0}\right) + \frac{D}{Q}\overline{\pi}\right]\sigma\sqrt{L}[1 - \Phi(k)] \tag{6}$$

$$\frac{TEAC^N}{\partial \pi_x} = \left[\frac{D}{Q}\left(\frac{2\beta_0\pi_x}{\pi_0} - \beta_0\right) - \frac{h\beta_0}{\pi_0}\right]\sigma\sqrt{L}\psi(k) \tag{7}$$

$$\frac{TEAC^N}{\partial A} = -\frac{\alpha B}{A} + \frac{D}{Q} \tag{8}$$

$$\frac{TEAC^N}{\partial L} = \frac{hk\sigma L^{-1/2}}{2} + \frac{\sigma L^{-1/2}\psi(k)}{2}\left[h\left(1 - \frac{\beta_o\pi_x}{\pi_0}\right) + \frac{D}{Q}\overline{\pi}\right] - \frac{D}{Q}m_i \tag{9}$$

To obtain the global minimum solution of this model, the following second order partial derivatives as in

(10)-(13) are used to calculate all minors as follows,

$$\frac{\partial^2 TEAC^N}{\partial Q^2} = \frac{2AD}{Q^3} + \frac{2D\overline{\pi}\sigma\sqrt{L}\psi(k)}{Q^3} + \frac{2D}{Q^3}C(L) \tag{10}$$

$$\frac{\partial^2 TEAC^N}{\partial k^2} = \left[h\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)+\frac{D}{Q}\overline{\pi}\right]\sigma\sqrt{L}\phi(k) \tag{11}$$

$$\frac{\partial^2 TEAC^N}{\partial \pi_x^2} = \left[\frac{D}{Q}\left(\frac{2\beta_0}{\pi_o}\right)\right]\sigma\sqrt{L}\psi(k) \tag{12}$$

$$\frac{\partial^2 TEAC^N}{\partial A^2} = \frac{\alpha B}{A^2} \tag{13}$$

Now, it is clear that, for fixed (A,Q,k,π_x), $TEAC^N(A,Q,k,\pi_x,L)$ is concave in L, which is given in (14).

$$\frac{\partial^2 TEAC^N}{\partial L^2} = -\frac{1}{4}hk\sigma L^{\frac{-3}{2}} - \frac{1}{4}\left[h\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)+\frac{D}{Q}\overline{\pi}\right]\sigma L^{\frac{-3}{2}}\psi(k) < 0 \tag{14}$$

Therefore for fixed (A, Q, K, π_x), the minimum total expected annual cost will occur at the end points of the internal $[L_i, L_{i-1}]$. On the other hand, for fixed L $\in [L_i, L_{i-1}]$ by equating (5)-(8) to zero, one can obtain,

$$Q = \sqrt{\frac{2D[A+\overline{\pi}\sigma\sqrt{L}\psi(k)+C(L)]}{h}} \tag{15}$$

$$\Phi(k) = 1 - \frac{Qh}{hQ\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)+D\overline{\pi}} \tag{16}$$

$$\pi_x = \frac{hQ}{2D} + \frac{\pi_0}{2} \tag{17}$$

$$A = \frac{\alpha BQ}{D} \tag{18}$$

Therefore, for fixed $L \in [L_i, L_{i-1}]$, $0 < A \leq A_0$ and $0 \leq \pi_x \leq_{\pi_0}$ are ignored, from (15)-(18), near optimal values of Q, k, π_x, A (we denote these values by Q^*, K^*, π_x^*, A^*) are obtained that the total expected annual cost is minimum.

Proposition 1

For fixed $L \in [L_i, L_{i-1}]$, the Hessian matrix of $TEAC^N$(A,Q,k,π_x,L) is positive definite at the point (Q^*, K^*, π_x^*, A^*) as obtained from (15)-(18).

Proof of proposition 1

For given value of L, the Hessian matrix H is,

$$H = \begin{bmatrix} \frac{\partial^2 TEAC^N(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 TEAC^N(\cdot)}{\partial Q^*\partial \pi_x{}^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial Q^*\partial k^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial Q^*\partial A^*} \\ \frac{\partial^2 TEAC^N(\cdot)}{\partial \pi_x{}^*\partial Q^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial \pi_x{}^{*2}} & \frac{\partial^2 TEAC^N(\cdot)}{\partial \pi_x{}^*\partial k^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial \pi_x{}^*\partial A^*} \\ \frac{\partial^2 TEAC^N(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial k^*\partial \pi_x{}^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial k^{*2}} & \frac{\partial^2 TEAC^N(\cdot)}{\partial k^*\partial A^*} \\ \frac{\partial^2 TEAC^N(\cdot)}{\partial A^*\partial Q^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial A^*\partial \pi_x{}^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial A^*\partial k^*} & \frac{\partial^2 TEAC^N(\cdot)}{\partial A^{*2}} \end{bmatrix}$$

where $TEAC^N(\cdot) = TEAC^N(Q^*, K^*, \pi_x^*, L^* A^*)$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial {Q^*}^2} = \frac{2A^*D}{{Q^*}^3} + \frac{2D\overline{\pi}\sigma\sqrt{L}\psi(k^*)}{{Q^*}^3} + \frac{2DC(L)}{{Q^*}^3}$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial {\pi_x^*}^2} = \frac{2D\beta_0}{Q^*\pi_0}\sigma\sqrt{L}\psi(k^*)$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial {k^*}^2} = \left[\frac{D}{Q^*}\overline{\pi} + h\left(1 - \frac{\beta_0 {\pi_x}^*}{\pi_0}\right)\right]\sigma\sqrt{L}\phi(k^*)$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial {A^*}^2} = \frac{\alpha B}{{A^*}^2}$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial Q^* \partial {\pi_x}^*} = \frac{\partial^2 TEAC^N(\cdot)}{\partial {\pi_x}^* \partial Q^*} = \frac{D\sigma\sqrt{L}\psi(k^*)}{{Q^*}^2}\left(\beta_0 - \frac{2\beta_0 {\pi_x}^*}{\pi_0}\right)$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial Q^* \partial k^*} = \frac{\partial^2 TEAC^N(\cdot)}{\partial k^* \partial Q^*} = \frac{D\overline{\pi}\sigma\sqrt{L}}{{Q^*}^2}[1 - \Phi(k^*)]$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial Q^* \partial A^*} = \frac{\partial^2 TEAC^N(\cdot)}{\partial A^* \partial Q^*} = -\frac{D}{{Q^*}^2}$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial k^* \partial \pi_{x^*}} = \frac{\partial^2 TEAC^N(\cdot)}{\partial \pi_{x^*} \partial k^*} = \left[\frac{h\beta_0}{\pi_0} - \frac{D}{Q^*}\left(\frac{2\beta_0 \pi_{x^*}}{\pi_0} - \beta_0\right)\right]\sigma\sqrt{L}[1 - \Phi(k^*)]$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial k^* \partial A^*} = \frac{\partial^2 TEAC^N(\cdot)}{\partial A^* \partial k^*} = 0$$

$$\frac{\partial^2 TEAC^N(\cdot)}{\partial {\pi_x}^* \partial A^*} = \frac{\partial^2 TEAC^N(\cdot)}{\partial A^* \partial {\pi_x}^*} = 0$$

The first principal minor of H is,

$$|H_{11}| = \frac{2A^*D}{{Q^*}^3} + \frac{2D\overline{\pi}\sigma\sqrt{L}\psi(k^*)}{{Q^*}^3} + \frac{2DC(L)}{{Q^*}^3} > 0$$

The second principal minor of H is,

$$|H_{22}| = \left[\frac{2A^*D}{{Q^*}^3} + \frac{2D\overline{\pi}\sigma\sqrt{L}\psi(k^*)}{{Q^*}^3} + \frac{2DC(L)}{{Q^*}^3}\right]\left[\frac{2D\beta_0}{Q^*\pi_0}\sigma\sqrt{L}\psi(k^*)\right] - \left\{\frac{D^2}{{Q^*}^4}\left(\frac{2\beta_0 {\pi_x}^*}{\pi_0} + \beta_0\right)^2 [\sigma\sqrt{L}\psi(k^*)]^2\right\}$$

$$= \frac{4D^2\beta_0\sigma\sqrt{L}\psi(k^*)}{{Q^*}^4\pi_0}[A^* + C(L)] + \frac{D^2\beta_0[\sigma\sqrt{L}\psi(k^*)]^2}{{Q^*}^4}(4 - \beta_0) > 0$$

The third principal minor of H is,

$$|H_{33}| = \left\{\frac{4D^2\beta_0\sigma\sqrt{L}\psi(k^*)}{{Q^*}^4\pi_0}[A^* + C(L)] + \frac{D^2\beta_0[\sigma\sqrt{L}\psi(k^*)]^2}{{Q^*}^4}(4 - \beta_0)\right\}\left[\frac{D}{Q^*}\overline{\pi} + h\left(1 - \frac{\beta_0 {\pi_x}^*}{\pi_0}\right)\right]\sigma\sqrt{L}\phi(k^*)$$

$$-\frac{2D\beta_0}{Q^*\pi_0}\sigma\sqrt{L}\psi(k^*)\left[\frac{D\overline{\pi}\sigma\sqrt{L}}{{Q^*}^2}\{1 - \Phi(k^*)\}\right]^2$$

$$= \frac{4D^2\beta_0\sigma\sqrt{L}\psi(k^*)}{{Q^*}^4\pi_0}[A^* + C(L)]\left\{\frac{D}{Q^*}\overline{\pi} + h\left(1 - \frac{\beta_0 {\pi_x}^*}{\pi_0}\right)\right\}\sigma\sqrt{L}\phi(k^*) +$$

$$\frac{D^2\beta_0\left\{\sigma\sqrt{L}\psi(k^*)\right\}^2}{Q^{*^4}}(4-\beta_0)h\left(1-\frac{\beta_0\pi_x{}^*}{\pi_0}\right)\sigma\sqrt{L}\phi(k^*)$$

$$+\frac{D^2\beta_0\left\{\sigma\sqrt{L}\psi(k^*)\right\}^2}{Q^{*^4}}(4-\beta_0)\frac{D}{Q^*}\overline{\pi}\sigma\sqrt{L}\phi(k^*)$$

$$-\frac{2D\beta_0}{Q^*\pi_0}\sigma\sqrt{L}\psi(k^*)\left[\frac{D^2\overline{\pi}^2\sigma^2L}{Q^{*^4}}\left\{1-\Phi(k^*)\right\}^2\right]$$

$$>\frac{D^3\beta_0\left\{\sigma\sqrt{L}\psi(k^*)\right\}^2}{Q^{*^5}}(4-\beta_0)\overline{\pi}\sigma\sqrt{L}\phi(k^*)-\frac{\beta_0\sigma\sqrt{L}\phi(k^*)}{\pi_0}\left[\frac{2D^3\overline{\pi}^2\sigma^2L}{Q^{*^5}}\left\{1-\Phi(k^*)\right\}^2\right]$$

$$=\frac{D^3\sigma^2L}{Q^{*^5}}\sigma\sqrt{L}\psi(k^*)\overline{\pi}\left[\beta_0\psi(k^*)(4-\beta_0)\phi(k^*)-2\frac{\beta_0}{\pi_0}\left\{1-\Phi(k^*)\right\}^2\overline{\pi}\right]$$

$$=\frac{D^3\sigma^2L}{Q^{*^5}}\sigma\sqrt{L}\psi(k^*)\overline{\pi}\left[\beta_0\psi(k^*)(4-\beta_0)\phi(k^*)-2(1-\Phi(k^*))^2\left\{\frac{\beta_0{}^2\pi_x{}^*}{\pi_0{}^2}(\pi_x{}^*-\pi_0)+\beta_0\right\}\right]$$

$$>\frac{D^3\sigma^2L}{Q^{*^5}}\sigma\sqrt{L}\psi(k^*)\overline{\pi}\left[\beta_0\psi(k^*)(4-\beta_0)\phi(k^*)-2\beta_0(1-\Phi(k^*))^2\right]$$

$$=\frac{D^3\sigma^2L}{Q^{*^5}}\sigma\sqrt{L}\psi(k^*)\overline{\pi}\left[\beta_0(2-\beta_0)\psi(k^*)\phi(k^*)+2\beta_0\psi(k^*)\phi(k^*)-2\beta_0(1-\Phi(k^*))^2\right]$$

$$>\frac{2D^3\sigma^2L\beta_0}{Q^{*^5}}\sigma\sqrt{L}\psi(k^*)\overline{\pi}\left[\psi(k^*)\phi(k^*)-(1-\Phi(k^*))^2\right]>0$$

The fourth principal minor determinant of H is,

$$|H_{44}|=\frac{\alpha B}{A^{*^2}}|H_{33}|$$

$$+\frac{D}{Q^{*^2}}\left[-\frac{2D\beta_0\sigma\sqrt{L}\psi(k^*)}{Q^*\pi_0}\frac{D}{Q^{*^2}}\left\{\frac{D\overline{\pi}}{Q^*}+h\left(1-\frac{\beta_0\pi_x{}^*}{\pi_0}\right)\right\}\sigma\sqrt{L}\phi(k^*)\right.$$
$$\left.+\left\{\frac{h\beta_0}{\pi_0}-\frac{D}{Q^*}\left(\frac{2\beta_0\pi_x{}^*}{\pi_0}-\beta_0\right)\right\}^2(\sigma\sqrt{L}[1-\Phi(k^*)])^2\frac{D}{Q^{*^2}}\right]$$

$$=\frac{\alpha B}{A^{*^2}}|H_{33}|+\left(\frac{D}{Q^{*^2}}\right)^2\left\{\frac{h\beta_0}{\pi_0}-\frac{D}{Q^*}\left(\frac{2\beta_0\pi_x{}^*}{\pi_0}-\beta_0\right)\right\}^2(\sigma\sqrt{L}[1-\Phi(k^*)])^2$$
$$-\left(\frac{D}{Q^{*^2}}\right)^2\left[\frac{2D\beta_0\sigma\sqrt{L}\psi(k^*)}{Q^*\pi_0}\left\{\frac{D\overline{\pi}}{Q^*}+h\left(1-\frac{\beta_0\pi_x{}^*}{\pi_0}\right)\right\}\right]\sigma\sqrt{L}\Phi(k^*)$$

$$=\frac{\alpha B}{A^{*^2}}|H_{33}|+\left(\frac{D}{Q^{*^2}}\right)^2\left[\left\{\frac{h\beta_0}{\pi_0}-\frac{D}{Q^*}\left(\frac{2\beta_0\pi_x{}^*}{\pi_0}-\beta_0\right)\right\}^2(\sigma\sqrt{L}[1-\Phi(k^*)])^2\right.$$
$$\left.-\frac{2D\beta_0\sigma\sqrt{L}\psi(k^*)\sigma\sqrt{L}\phi(k^*)}{Q^*\pi_0}\left\{\frac{D\overline{\pi}}{Q^*}+h\left(1-\frac{\beta_0\pi_x{}^*}{\pi_0}\right)\right\}\right]$$

$$=\frac{\alpha B}{A^{*^2}}|H_{33}|+\left(\frac{D}{Q^{*^2}}\right)^2\left[\left\{\frac{h\beta_0}{\pi_0}-\frac{D\beta_0}{Q^*\pi_0}(2\pi_x{}^*-\beta_0)\right\}^2(\sigma\sqrt{L}[1-\Phi(k^*)])^2-\frac{2D\beta_0\sigma^2L\psi(k^*)\phi(k^*)h}{Q^*\pi_0(1-\Phi(k^*))}\right]$$

$$=\frac{\alpha B}{A^{*^2}}|H_{33}|+\left(\frac{D}{Q^{*^2}}\right)^2\left[\left\{\frac{\beta_0}{\pi_0Q^*}\left(hQ^*-2D\left(\pi_x{}^*-\frac{\pi_0}{2}\right)\right)\right\}^2(\sigma\sqrt{L}[1-\Phi(k^*)])^2-\frac{2D\beta_0\sigma^2L\psi(k^*)\phi(k^*)h}{Q^*\pi_0(1-\Phi(k^*))}\right]$$

$$> \frac{\alpha B}{A^{*2}}|H_{33}| - \left(\frac{D}{Q^{*2}}\right)^2 \frac{2D\beta_0\sigma^2 L\psi(k^*)\phi(k^*)h}{Q^*\pi_0(1-\Phi(k^*))}$$

$$= \frac{\alpha B}{A^{*2}}|H_{33}| + \left(\frac{D}{Q^{*2}}\right)^2 \frac{2D\beta_0\sigma^2 L\psi(k^*)\phi(k^*)h}{Q^*\pi_0(\Phi(k^*)-1)} > 0$$

As the principal minors of the Hessian matrix are positive, the Hessian matrix H is positive definite at the point (Q^*, K^*, π_x^*, A^*). Hence the total expected annual cost function is a global minimum at that point.

Now two constraints $0 < A \leq A_0$ and $0 \leq \pi_x \leq \pi_0$ are considered. As α, B, Q, D, h, and π_0 are all positive, it is clear from (17) and (18) that $\pi_x{}^*$and A^* are positive. If $A^* > A_0$, then no investment should be made for the reduction in setup cost, and if $\pi_x{}^* > \pi_0$, then no discount will be offered by supplier. Therefore, it would be considered $A^* = A_0$ and $\pi_x{}^* = \pi_0$. For this type of problem, it is not possible to obtain closed-form solutions and for lead time, the cost function is concave. Thus, the solutions, as near optimal solutions rather than as a global optimum are obtained. An algorithm is implemented to obtain the near optimal values of Q, k, π_x, A, and L.

3.3. KSS algorithm

Step 1 For each $L_{i,i=1,2,\ldots,n}$, perform 1a to 1e.

1a Set $\pi_{xi1} = \pi_0, A_{i1} = A_0$ and $k_{i1} = 0$ (implies $\psi(k_{i1})$=0.39894).

1b Substituting π_{xi1}, A_{i1} and $\psi(k_{i1})$ into (15), evaluate Q_{i1}.

1c Utilizing Q_{i1}, obtain $\Phi(k_{i2})$ from (16); hence, find (k_{i2}) by checking the normal table and evaluate $\psi(k_{i2})$.

1d Utilizing Q_{i1}, determine π_{xi2} and A_{i2} from (17) and (18), respectively.

1e Repeat 1a to 1d until no changes occur in the values of Q_i, k_i, π_{xi} and A_i. Denote the solution by $(\widehat{Q}_i, \widehat{k}_i, \widehat{\pi_{xi}}, \widehat{A}_i)$

Step 2 For each i=1, 2,..., n, compare $\widehat{\pi_{xi}}$ with π_0 and $\widehat{A}_i$ with A_0.

2a If $\widehat{\pi_{xi}} < \pi_0$ and$\widehat{A}_i < A_0$, then the solution found in step 1 is the near optimal for given L_i. Denote this solution by (Q^*, K^*, π_x^*, A^*). Go to Step 4.

2b If $\widehat{\pi_{xi}} \geq \pi_0$ and $\widehat{A}_i$<A_0, then for given L_i, set $\pi_{xi}{}^* = \pi_0$ and evalute new $(\widehat{Q}_i, \widehat{k}_i, \widehat{A}_i)$ from (15), (16), and (18) by the same procedure as described in step 1. If $\widehat{A}_i < A_0$, then the near optimal solution is (Q^*, K^*, π_x^*, A^*) = $(\widehat{Q}_i, \widehat{k}_i, \pi_0, \widehat{A}_i)$, and go to step 4. Otherwise, go to step 3.

2c If $\widehat{A}_i \geq A_0$ and $\widehat{\pi_{xi}} < \pi_0$, then for given L_i, set $A^* = A_0$ and evaluate new $(\widehat{Q}_i, \widehat{k}_i, \widehat{\pi_{xi}})$ from (15), (16), and (17) by the same procedure as described in step 1. If $\widehat{\pi_{xi}} < \pi_0$ then the near optimal solution is (Q^*, K^*, π_x^*, A^*) = $(\widehat{Q}_i, \widehat{k}_i, \pi_0, \widehat{A}_i)$, and go to step 4. Otherwise, go to 3.

2d If $\widehat{\pi_{xi}} \geq \pi_0$ and $\widehat{A}_i \geq A_0$, go to step 3.

Step 3 For the given L_i, set $\pi_{xi}{}^* = \pi_0$ and $A_i{}^* = A_0$ using (15) and (16) to obtain the corresponding near optimal solution $(Q_i{}^*, k_i{}^*)$ using Step 1.

Step 4 Utilize (4) to calculate the corresponding total expected cost $TEAC^N(Q_i{}^*, k_i{}^*, \pi_{xi}{}^*, A_i{}^*, L_i)$.

Step 5 $TEAC^N(Q^*, k^*, \pi_x{}^*, A^*, L) = min_{i=1,2,\ldots,n} TEAC^N(Q_i{}^*, k_i{}^*, \pi_{xi}{}^*, A_i{}^*, L_i)$ gives $(Q^*, k^*, \pi_x{}^*, A^*, L)$, the near optimal solution. After substituting the values of k^* and L^*, the reorder point can be obtained as $r^* = DL + k^*\sigma\sqrt{L^*}$

4. NUMERICAL EXAMPLES

To illustrate the above problem, the parametric values are considered as follows: D=600 units/year, A_0=\$200/order, h=\$20/unit/year, π_0=\$150/unit, σ=7units, α=\$0.1/year, and B=5800, and the lead time has three components with data, as shown in table 2.

Table 2.Lead time data

Lead time component i	Normal duration v_i(days)	Minimum duration u_i(days)	Unit crashing cost m_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Example This example deals with a model having lead time demand that follows the normal distribution. The results are obtained for the upper bounds of the backorder rate β_0= 0.0, 0.5, 0.8, 1.0. Using KSS Algorithm, the near optimal results are tabulated in table 3.

Table 3.The near optimal solution for normal distribution model (Li in weeks)

β_0	Investment and price discount (proposed model)		No investment and no price discount(A=$A_0 = 200$ and $\pi_x = \pi_0 = 150$)		SV
	$(A^*, Q^*, \pi_x{}^*, k^*, L^*)$	$TEAC_2^N(\cdot)$	(Q^*, k^*, L^*)	$TEAC_1^N(\cdot)$	
0.0	(81.18, 83.98, 76.40, 2.09, 4)	2789.57	(120.81, 1.94, 4)	2962.44	5.84
0.5	(81.34, 84.15, 76.40, 2.03, 4)	2775.60	(120.89, 1.93, 4)	2961.03	6.26
0.8	(81.46, 84.27, 76.40, 2.00, 4)	2766.06	(120.94, 1.93, 4)	2960.18	6.56
1.0	(81.55, 84.36, 76.41, 1.96, 4)	2759.11	(120.98, 1.93, 4)	2959.61	6.77

$$Note : SV = \{TEAC_1^N(\cdot) - TEAC_2^N(\cdot)/TEAC_1^N(\cdot)\} * 100\%$$

Now the numerical result is compared with [24] model in order to examine the effects of considering the setup

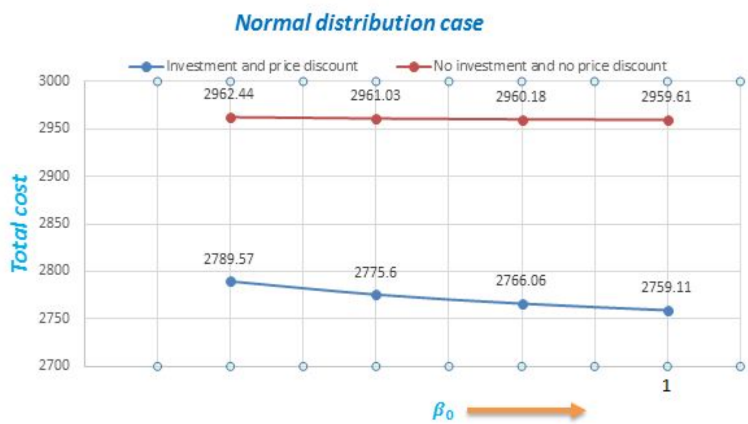

Figure 1.Graphical representation of total cost for distinct value of β_0

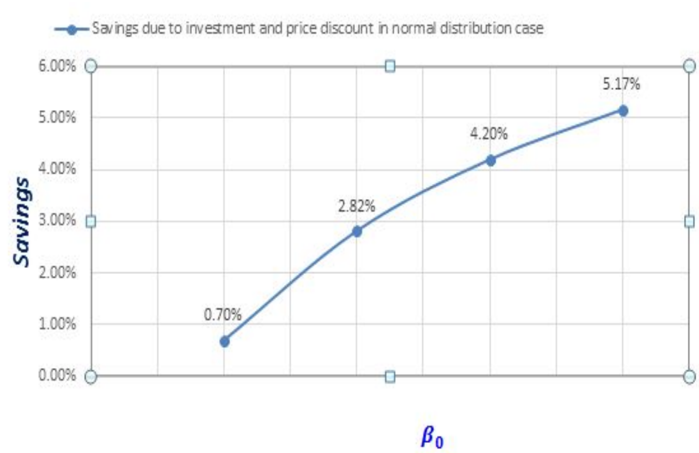

Figure 2.Graphical representation of savings for distinct value of β_0

cost reduction and backorder price-discount in this model. The summarization of these comparisons is shown in table 4. From table 4, we can see that our model obtained a better result than [24] model.

Sensitivity analysis

The effects of changes in parameters such as A_0, h, α, σ and B on the total cost are studied. The sensitivity analysis is performed by changing each of the parameters by $-50\%, -25\%, +25\%$ and $+50\%$ taking one parameter

Table 4.Numerical comparison with [24] model

β_0	Normal distribution model		
	Total cost		
	Our model	[24]	Savings %
0.0	2759.11	3114.36	11.41
0.5	2755.60	3098.57	10.42
0.8	2766.06	3084.36	10.31
1.0	2759.11	3073.98	10.24

at a time while keeping the remaining parameters unchanged. The results of example is presented in table 5.

Table 5.Sensitivity analysis for key parameters

Parameters	Changes (in%)	Normal distribution model
A_0	-50%	2364.04
	-25%	2599.21
	0%	2766.06
	+25%	2895.49
	+50%	3001.23
h	-50%	1943.03
	-25%	2391.31
	0%	2766.06
	+25%	3099.01
	+50%	3404.51
A	-50%	2371.22
	-25%	2605.001
	0%	2766.066
	+25%	2869.25
	+50%	2924.02
σ	-50%	2433.25
	-25%	2600.00
	0%	2766.06
	+25%	2931.46
	50%	3096.19
B	-50%	2371.22
	-25%	2605.00
	0%	2766.06
	+25%	2869.25
	+50%	2924.02

From table 5, the discussion of sensitivity analysis of the key parameters is as follows:

1. If the setup cost increases, then the total cost also increases, but there is an investment to reduce setup cost, thus table 6 indicates that 50% change in setup cost gives only 30% change in total cost function as in figure 3.
2. Increasing the value of the holding cost increases the total cost. From table 5, it is observed that a negative change in holding cost results in greater reduction in total cost than does a positive change. Thus, it can be concluded that the holding cost is more sensitive to negative change than positive change with regard to total cost. This is the most sensitive cost parameter in this proposed model. The important observation is that the change in total cost with holding cost is greater in the case of distribution free approach as in figure 4.
3. If the annual fractional cost of capital investment is increased, then the total cost is also increased. This is a more sensitive parameter in normal distribution model, whereas it is less sensitive in distribution free model as in figure 5.
4. If the standard deviation of lead time demand increases while all the other parameters remain unchanged, the expected total cost tends to increase. The negative and positive changes in the standard deviation give approximately the same amount of change in total cost function as in figure 6.

5. Increasing the value of parameter B increases the total cost. From table 5, it is easy to see that this parameter is more sensitive in normal distribution model than distribution free model as in figure 7.

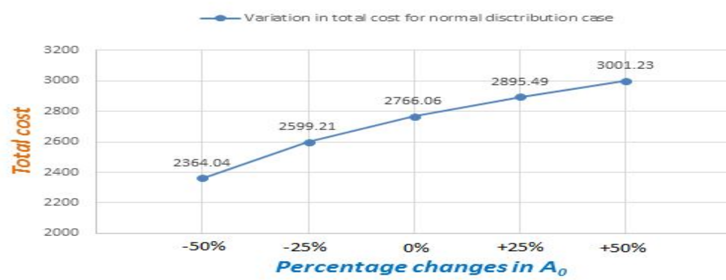

Figure 3.Effect of percentage changes in A_0 on total cost

Figure 4.Effect of percentage changes in h on total cost

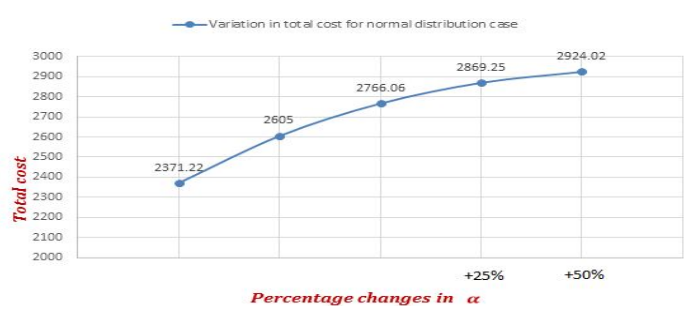

Figure 5.Effect of percentage changes in α on total cost

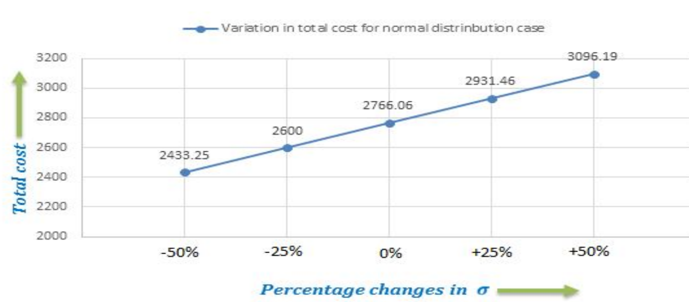

Figure 6.Effect of percentage changes in σ on total cost

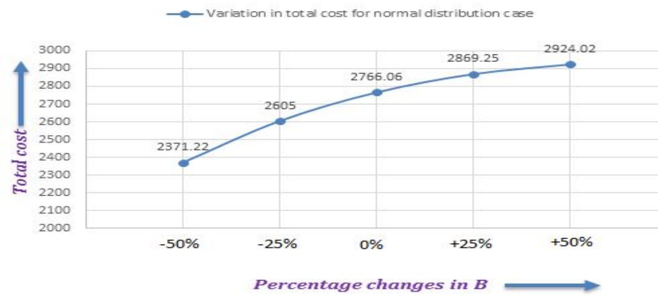

Figure 7.Effect of percentage changes in B on total cost

5. CONCLUSIONS

The model investigates the effect of setup cost reduction with order quantity, setup cost, reorder point, backorder price discount rate and lead time as decision variables. The model is proposed with a normally distributed lead time demand. For fixed lead time, it is possible to obtain the global optimal solution by using a proposition but including variable lead time, it is near optimal solution as the sufficient condition for lead time is not satisfied. A logarithmic investment function is considered to reduce the total setup cost. In order to secure more backorder during shortage, this model allowed price-discount policy on stock out items for customers. This price-discount offer makes the customers more willing to wait for the desired items. Moreover, one can observe from the numerical example that as the upper bound of backorder price-discount increases, the total cost of the inventory system decreases. The numerical results indicate that if someone make decisions to investment in reducing setup cost and offering a backorder price-discount on stock out items to his customers, it will help him to reduce the system cost, and he can gain a significant amount of savings to increase the competitive edge in business. This model may be applied to such condition where stock out situations occur and some customers are wiling for backorder. There are several strategies for the managers. They have to take decision based on setup cost reduction. If they have the ability to invest, they can use the proposed policy to save some funds by using the setup cost reduction policy. The marketing environment is continuously changing, thus they must be adjusted with more savings and justified policies. The manger can decide based on the expected value of additional information about lead time demand distribution that the manager would like to consider normal distribution or distribution free approach to calculate expected shortages during stock out situation. For future research, the model can be extended to improve the quality of products. Another interesting research may be conducted by considering inspection error in an inspection policy with imperfect production. One can consider multi-item and permissible delay-in-payments with uncertain received quantity.

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