DJ Journal of Engineering and Applied Mathematics, Vol. 4(2) 2018, pp. 21-33


RESEARCH ARTICLE

# Numerical Method for Non-Stiff Differential Equations

*K Selvakumar[1]
[1]Department of Mathematics & University College of Engineering, Nagercoil, Anna University, Tamil Nadu, India.

Received- 23 January 2018, Revised- 20 February 2018, Accepted- 1 March 2018, Published- 3 April 2018

**ABSTRACT**

The path of the trajectory of an aircraft from ground to space is modeled mathematically as a solution of the non-stiff differential equation. When the exact path is not tracable, we have to go for numerical solution of the model. To give optimal guidance to the aircraft, the path of the trajectory must be known. The need for the numerical method is to find the solution of a differential equation when the exact solution is not known. Traditionally, the order of convergence of the solution is proved theoretically and it can be tested with a test problem, if the exact solution is known. When the exact solution is not known, one cannot test the order of convergence of the numerical solution using the proved error estimates. In this paper, Euler method is applied to solve non-stiff differential equation when it does not have exact solution in hand, and an error estimate is derived to check the order of convergence of the solution of the numerical method. To show the behaviour of the numerical solution such as stability and order, in the complex plane, stability regions, order star, order star finger region, relative stability region, relative absolute region and order graph are presented. Experimental results are presented to show the performance of the numerical method with respect to the metrics such as regions of stability and theoretical and numerical rate of order of convergence of the solution, both numerically and graphically using MATLAB.
**Keywords:** Aircraft trajectory path modelling, Non-stiff differential equation, Euler's explicit method, Absolute error, Stability region.

## 1. INTRODUCTION

Mathematical modelling of the aircraft trajectory path from ground to space is computed as a solution of the non-stiff differential equation. Our problem to solve the non-stiff ordinary differential equation with initial condition is,

$$y'(t) = f(t, y(t)), t \in [a,b], y(a) = \phi \tag{1}$$

which is subjected to the stability condition $f_y(t, y(t)) \geq \alpha > 0$ for all $t \in [a,b]$. The assumption made here is only to consider stable path of the trajectory of the aircraft. This equation is also a mathematical model for growth of a plant from ground to space.

Problem (1) is a problem of finding the shape of the unknown curve which starts at a given point and satisfies the given ordinary non-stiff differential equation. In 1769-70, [1], Leonhard Euler in his book, Institutionum Calculi Integralis, designed a numerical method for the solution of the problem (1) and it is of the form,

$$y_{i+1} = y_i + hf(t_i, y_i), i = 0, 1, 2, \ldots.., N-1, y_0 = \phi \tag{2}$$

where $a = t_0, t_1, t_2, \ldots, t_N = b$ is a sequence of points in [a, b], using the step size h, $t_i = t_0 + ih$ and $h = t_{i+1} - t_i, i = 0, 1, 2, \ldots, N-1$. The method (2) is consistent with the problem (1) as step size h approaches zero ($h \to 0$).

When the exact solution of a problem is not known, one must go for a numerical solution. Traditionally, the error estimate derived can be tested for validity when the exact solution of the problem is known. When the exact solution is not known, one cannot test for validity, and this motivates to find the error estimates of the absolute error, so that one can check for the rate of order of convergence of the numerical solution directly.

We consider the numerical method of Euler (2) only in this paper. The procedure followed in this paper is new. The stability regions for non-stiff problems using Euler's method is clearly presented in this paper.

*Corresponding author. Tel.: +919566604812
Email address: k_selvakumar10@yahoo.com (K.Selvakumar)
Double blind peer review under responsibility of DJ Publications
https://dx.doi.org/10.18831/djmaths.org/2018021003

Numerical methods have been generated from Euler's explicit method in the works of the author [2-19]. [20, 21] A complete survey of the numerical methods for ordinary non-stiff differential equation is given and the development of numerical solution techniques from the identification of the problem to the never-final preparation of automatic codes for the solution of classes of similar problems is examined. [22] has applied the numerical methods for the problems in NASA. The students of bachelors degree in Computer Sciences or Industrial Engineering at the University of Salamanca, Spain, have solved real problems using numerical methods with the help of the already acquired interdisciplinary knowledge [23] in MATLAB. Scientists and electrical engineers must have clear picture about Euler's method, and with this aim this paper is written.

In section 2, stability analysis is given using amplificaion factor and in section 3, stability function, stability region and order star fingers for non-stiff problems are presented. In section 4, for the completeness of this paper, an estimate for the absolute error is derived, when the exact solution is known. In section 5, an error estimate for the absolute error is derived when exact solution is not known.

Finally, in section 6, the experimental results are presented to show the performance of the method both numerically and graphically. Throughout this paper C is a constant independent of i and h.

## 2. STABILITY ANALYSIS

This section gives the stability analysis and order graph of the non-stiff problem using amplification factor. On applying the method (2) to the Dahlquist problem [24],

$$y' = \lambda y, t \in [0,1], \lambda > 0, y(0) = 1 \tag{3}$$

whose exact solution is $y(t) = exp(\lambda t)$. We have the difference equation as,

$$\frac{y_{i+1}}{y_i} = 1 + h\lambda \tag{4}$$

The amplification factor $Q(\lambda h)$ is defined as follows,

$$Q(\lambda h) = \frac{y_{i+1}}{y_i} \tag{5}$$

From (5) and (4) we have,

$$Q(\lambda h) = 1 + h\lambda \tag{6}$$

A numerical method is stable if,

$$|Q(\lambda h)| \leq 1 \tag{7}$$

Using (7), one can determine the maximum step size h for which the method is stable. Equation (7) holds only when $0 \leq h \leq \frac{2}{|\lambda|}$

## 3. STABILITY REGION

In this section, stability regions, order star, order star finger region, relative stability region and relative absolute region of the solution of non-stiff problem are presented using stability function, both theoretically and figuratively.

**Stability region of exact solution**: The stability function of the exact solution of the Dahlquist problem (3) is given by,

$$R(z) = exp(z) \tag{8}$$

where $z = h$, z is a complex number and the stability region is the entire left half of the complex plane which is presented in figure 1.

**Stability region of numerical solution**: On applying Euler's method (2) to the problem (3), the stability function of the numerical solution is given by,

$$R(z) = 1 + z \tag{9}$$

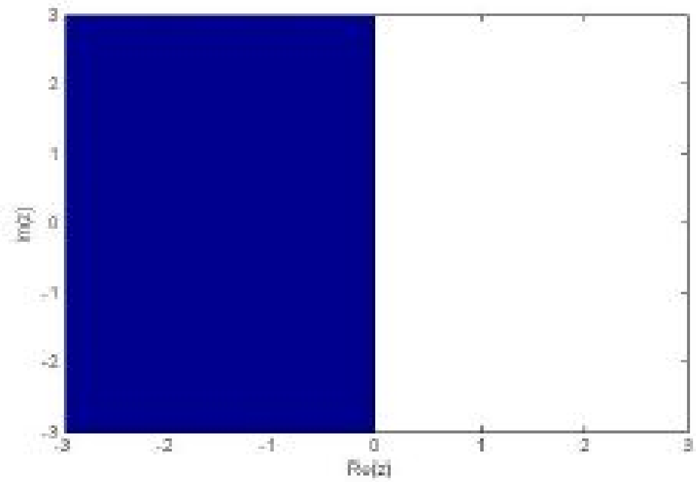


Figure 1.Stability region of exact solution

where $z = h\lambda$, z is a complex number and the stability region is a region inside unit circular region with center (-1, 0) in the left half complex plane which is presented in figure 2.

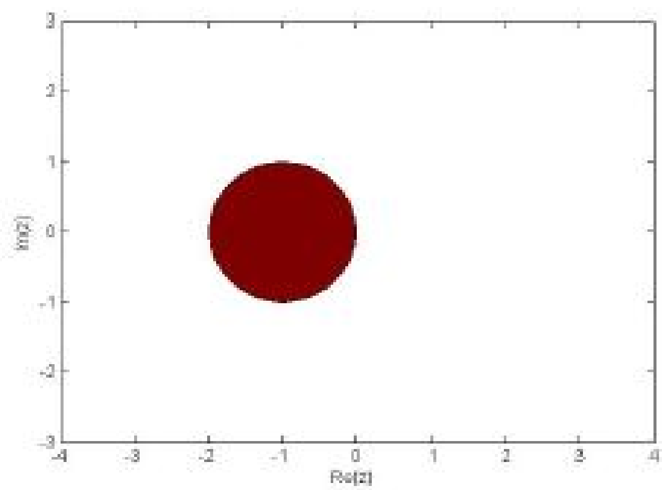


Figure 2.Stability region of numerical solution

On applying the Euler forward method to the problem (3), the numerical solution is stable inside the contour and unstable, if the values of $z = h\lambda$ is outside the region $z \in C/|1+z| \leq 1$

**Order graph**: The degree of the stability function of a numerical method is the order of the method. The degree of the stability function of the Euler's method is one, and so the order of the numerical method is one. The order graph for the solutions of the problem (3) with $\lambda = 1$ and the numerical method is given in figure 3. The curve with + sign refers to the exact solution and the curve with * sign refers to the Euler's method. It is a plot of h and Q(h) with respect to the exact solution of problem (3), Q(h)=exp(h) and with respect to the Euler's method Q(h)=1+h.

**Order star**: The order of a method is the number of fingers inside the stability region of the numerical method when the stability region intersect with Re(z)=0. And it is shown in figure 4. Only one finger is inside the stability region and so the method is of order one.

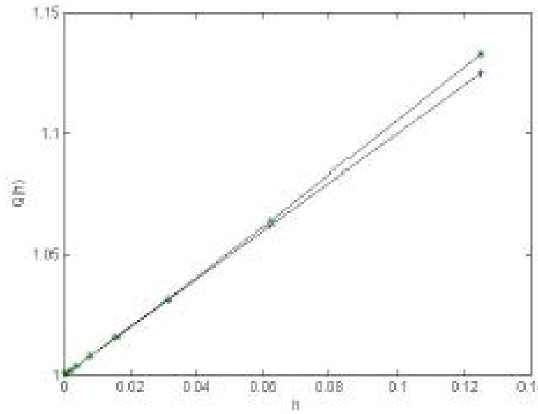


Figure 3.Order graph of Euler's forward method

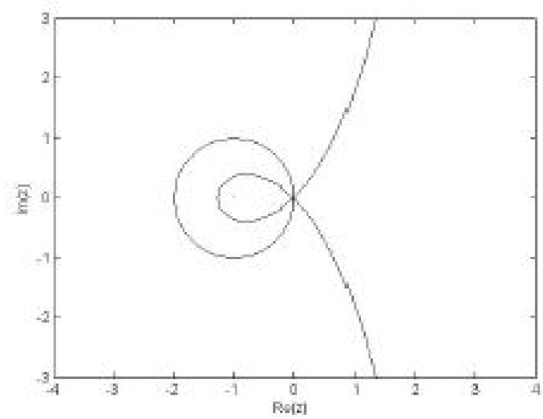


Figure 4.Order star-finger of numerical method

**Order star finger region of numerical method**: The order star region for Euler's forward method (2) is the region inside R(z)exp(-z) and it is given in figure 5.

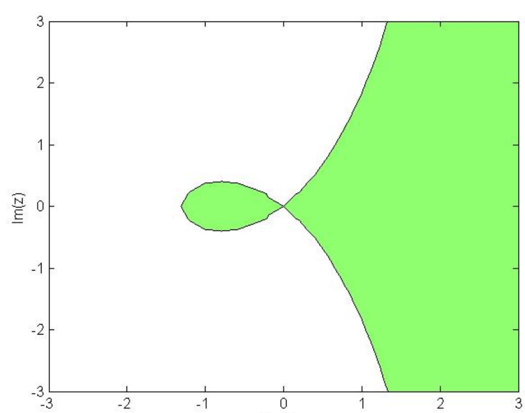


Figure 5.Order star-finger region of numerical method

**Relative stability region**: The relative stability region or the order star of first kind of the Euler's forward method is given in figure 6.

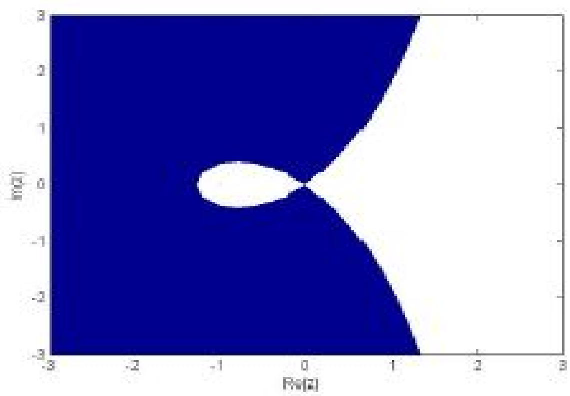


Figure 6.Relative stability region of numerical method

**Absolute relative stability region of numerical method**: Figure 7 shows the absolute stability region of the Euler forward method.

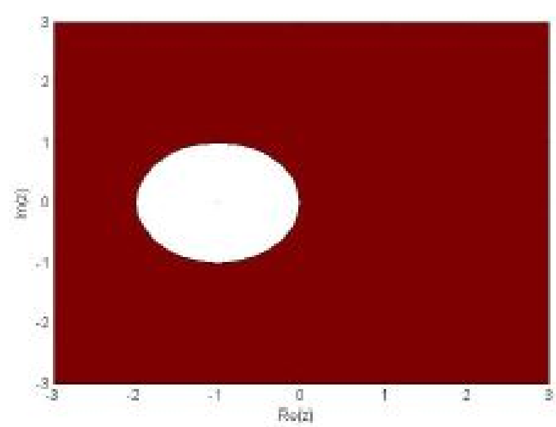


Figure 7.Absolute relative stability region of numerical method

## 4. THEORETICAL RATE OF ORDER OF CONVERBENCE

In this section, an error estimate of the theoretical absolute error for theoretical rate of order of convergence is proved to be of order one, so that one can check for the rate of order of convergence of the numerical solution directly with the help of a problem having exact solution in hand.

**Absolute error**: Absolute error is the difference between exact and approximate solutions at each time steps. The main result of this section is stated in the following theorem

**Theorem 4.1.** Let y(t) and $y_i$ be the solutions of the given differential equation (1) and the Euler's method (2) respectively. Then, for i=0(1)N, we have an error estimate of the form,

$$|y(t_i) - y_i| \leq Ch^1 \tag{10}$$

where C is independent of i and h.

**Proof**: The solution y(t) of (1) at the nodal point $t = t_{i+1}$ can be expressed in terms of solution y(t) at the nodal point at $t = t_i$, using Taylor's series as follows:

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + O(h^3), y(t_0) = \phi \tag{11}$$

where $h = t_{i+1} - t_i, i = 0, 1, ...N - 1$.

The numerical method (2) can be expressed as,

$$y_{i+1} = y_i + hf(t_i, y_i), y_0 = \phi \tag{12}$$

where $h = t_{i+1} - t_i$ and $y_i = y(t_i), i = 0, 1, ...N$.

(12) can be rewritten as,

$$y_{i+1} = y_i + hy'(t_i), y_0 = \phi \tag{13}$$

From (11) and (13), we have,

$$y(t_{i+1}) - y_{i+1} = y(t_i) - y_i + \frac{h^2}{2}y''(t_i) + O(h^3) \tag{14}$$

$$e_{i+1} = e_i + \frac{h^2}{2}y''(t_i) + O(h^3), e_0 = 0 \tag{15}$$

where $e_i = y(t_i) - y_i, i = 0, 1, ...N$. The error per step, Local Truncation Error (LTE) is given by,

$$LTE = \frac{h^2}{2}y''(t_i) + O(h^3) \tag{16}$$

The error at a given time t is termed as Global Truncation Error (GTE), and can be obtained from (16) by rewritting (16) as follows:

$$D_+e_i = \tfrac{h^1}{2}y''(t_i) + O(h^2)$$

$$GTE = \frac{h^1}{2}y''(t_i) + O(h^2) \tag{17}$$

From (17), we have to find the error estimate for the absolute error. For i=0,1,2,...,N-1, we have,

$$e_1 = e_0 + \frac{h^2}{2}y''(t_0) + O(h^3), e_0 = 0 \tag{18}$$

$$e_2 = e_1 + \frac{h^2}{2}y''(t_1) + O(h^3), ... \tag{19}$$

$$e_i = e_{i-1} + \frac{h^2}{2}y''(t_{i-1}) + O(h^3), ... \tag{20}$$

continuing like this finally,

$$e_N = e_{N-1} + \frac{h^2}{2}y''(t_{N-1}) + O(h^3) \tag{21}$$

From (18) to (21), we have,

$$e_N = \frac{h^2}{2}\sum_{j=0}^{N-1} y''(t_j) + O(h^3) \tag{22}$$

Let $k = max\left|y^{''}(t_j)\right| for j = 0(1)N-1$, then,

$$|e_N| \leq \frac{h^2}{2}\sum_{j=0}^{N-1}\left|y^{''}(t_j)\right| + O(h^3) \tag{23}$$

$$|e_N| \leq \frac{h^2}{2}\sum_{j=0}^{N-1} k + O(h^3)$$

$$|e_N| \leq \tfrac{h^2}{2}Nk + O(h^3)$$

since $N = \frac{b-a}{h}$ we have,

$$|e_N| \leq Ch \tag{24}$$

Now for i=0 and i=N we have the estimate. And for i=i,2,...N-1, we shall find the error estimate by taking (20)

$$e_i = \frac{h^2}{2}\sum_{m=0}^{i-1} y^{''}(t_m) + O(h^3)$$

$$|e_i| \leq \frac{h^2}{2}\sum_{m=0}^{i-1}\left|y^{''}(t_m)\right| + O(h^3)$$

since,

$$\sum_{m=0}^{i-1}\left|y^{''}(t_m)\right| \leq \sum_{j=0}^{N-1}\left|y^{''}(t_j)\right|$$

we have,

$$|e_i| \leq \frac{h^2}{2}\sum_{j=0}^{N-1}\left|y^{''}(t_j)\right| + O(h^3) \tag{25}$$

The right hand side of (25) is same as in (23), and so,

$$|e_i| \leq Ch \tag{26}$$

for i=1(1)N-1. Finally, from (18), (24) and (26), the desired estimate (10) follows for i=0(1)N

$$|e_i| \leq Ch \tag{27}$$

Hence the proof.

**Theoretical rate of order of convergence :** $y(t)$ is the solution of the given differential equation (1) and $y_i$ is the numerical solution of the numerical method. And, if we have an error estimate of the form,

$$|y(t_i) - y_i| \leq Ch^n \tag{28}$$

then n is the rate of order of convergence of the method and it can be obtained from the estimate (28). Rewriting (28) as,

$$e_i^h = Ch^n \text{ and } e_{2i}^{\frac{h}{2}} = C\tfrac{h^n}{2^n}$$

then taking the ratio, we get the rate of order of convergence of a numerical method as,

$$n = \frac{log(\frac{e_i^h}{e_{2i}^{\frac{h}{2}}})}{log2} \tag{29}$$

where $e_i^h = y(t_i) - y_i^h$ and $e_{2i}^{\frac{h}{2}} = y(t_{2i}) - y_{2i}^{\frac{h}{2}}$

Here $y_i^h$ stands for the numerical solution obtained by using step size h, and $y_{2i}^{\frac{h}{2}}$ stands for the numerical solution obtained by using step size $\frac{h}{2}$. From (10) and (29), the rate of order of convergence of Euler's method is of one (n=1).

It must be noted that in the estimates (10), (28) and (29), to find the theoretical rate of order of convergence, one must know the exact solution. The fact is, when the exact solution is not known, one must go for a numerical method to find solution. When exact solution is not known, finding out the rate of order of convergence is the next task. As a solution to this, an alternate error estimate for absolute error is derived and thereby the numerical rate of order of convergence is obtained.

## 5. NUMERICAL RATE OF ORDER OF CONVERGENCE

In this section, an error estimate of the numerical absolute error for numerical rate of order of convergence is proved to be of order one, and so the rate of order of convergence of the numerical solution can be checked directly with the help of a problem without exact solution in hand. The main result of this section is stated in the following theorem:

**Theorem 5.1.** Let $y(t)$ be the solution of the given differential equation (1) and $y_i^h$ and $y_{2i}^{\frac{h}{2}}$ be the numerical solutions of the Euler's method (2) using step sizes h and $\frac{h}{2}$ respectively. Then, for i=0(1)N, we have an error estimate of the form,

$$\left|y(t_i) - y_i^h\right| = \left|2[y_{2i}^{\frac{h}{2}} - y_i^h]\right| \leq Ch^1 \tag{30}$$

where $C$ is independent of $i$ and $h$.

**Proof**: Let $w_i = 2[y_{2i}^{\frac{h}{2}} - y_i^h]$ for i=9(1)N. For $i = 0$, $w_0 = 0$, and for i=1(1)N we have, from (2),

$$w_{i+1} = w_i + h[f(t_{2i}, y_{2i}^{\frac{h}{2}}) + f(t_{2i+1}, y_{2i+1}^{\frac{h}{2}})], w_0 = 0 \tag{31}$$

Using the procedure in theorem 4.1, (31) gets the form,

$$w_{i+1} = w_i + \frac{h^2}{2}y''(t_i) + O(h^3) \tag{32}$$

From (32), the error per step is given by,

$$LTE = \frac{h^2}{2}y''(t_i) + O(h^3) \tag{33}$$

Equation (33) can be rewritten as,

$$D_+w_i = \frac{h^1}{2}y''(t_i) + O(h^2)$$

and hence the error at a given time t is given by,

$$GTE = \frac{h^1}{2}y''(t_i) + O(h^2) \tag{34}$$

Now, equation (34) is for i=1(1)N. Adopting the procedure followed in theorm 4.1, for i=1(1)N, we have,

$$w_i = w_{i-1} + \frac{h^2}{2}y''(t_{i-1}) + O(h^3) \tag{35}$$

Now, for i=1(1)N, we have the relation,

$$w_i = \frac{h^2}{2}\sum_{m=0}^{i-1} y''(t_m) + O(h^3) \tag{36}$$

Following the procedure in theorm 4.1,

$$w_i \leq \frac{h^2}{2}\sum_{m=0}^{N-1} y''(t_m) + O(h^3) \tag{37}$$

Taking absolute value on both sides,

$$|w_i| \leq \frac{h^2}{2}\sum_{m=0}^{N-1} \left|y''(t_m)\right| + O(h^3) \tag{38}$$

Let $k = max\left|y''(t_m)\right|$, for m=0,1,2,...N-1, then, for i=1(1)N,

$$|w_i| \leq \frac{h^2}{2}\sum_{m=0}^{N-1} k + O(h^3) \leq \frac{h^2}{2}NK + O(h^3) \leq \frac{h^1}{2(b-a)}K + O(h^3) \tag{39}$$

From (31) and (39), we have the required result, for all i=0(1)N

$$|w_i| \leq Ch \tag{40}$$

Hence the proof.

**Numerical rate of order of convergence:** Let $y(t)$ be the solution of the given differential equation (1) and $y_i^h$ and $y_{2i}^{\frac{h}{2}}$ be the numerical solutions of the numerical method using step sizes h and $\frac{h}{2}$ respectively. Then, we have an error estimate of the form,

$$\left|y(t_i) - y_i^h\right| = \left|\frac{2^n}{2^n-1}[y_{2i}^{\frac{h}{2}} - y_i^h]\right| \leq Ch^n \tag{41}$$

Here n is the numerical rate of order of convergence of the method and it can be obtained from the estimate (41). Rewriting (41) as,

$w_i^h = Ch^n$ and $w_{2i}^{\frac{h}{2}} = C\frac{h^n}{2^n}$, and then taking the ratio, we get the numerical rate of order of convergence of a numerical method as,

$$n = \frac{log(\frac{w_i^h}{w_{2i}^{\frac{h}{2}}})}{log2} \tag{42}$$

where $w_i = 2[y_{2i}^{\frac{h}{2}} - y_i^h]$ and $w_{2i} = 2[y_{4i}^{\frac{h}{4}} - y_{2i}^{\frac{h}{2}}]$ Here $y_{2i}^{\frac{h}{2}}$ stands for the numerical solution obtained by using step size $\frac{h}{2}$, and $y_{4i}^{\frac{h}{4}}$ stands for the numerical solution obtained by using step size $\frac{h}{4}$. From (41) and (42), the numerical rate of order of convergence of Euler's method is of one (n=1).

## 6. EXPEIMENTAL RESULTS

In this section, both numerical and graphical results for the problem (3) with $\lambda = 1$ are given in the interval [0, 1].

**Experimental results with respect to order:** Following the error analysis in section 4, theorem 4.1 and expresion (34), in table 1, the theoretical rate of order of convergence and average rate of order of convergence are tabulated when the exact solution of the problem (3) with $\lambda = 1$ is known and exact solution is used for absolute error estimation. It is found that the theoretical rate of order of convergence and average theoretical rate of order of convergence are of one.

Following the error analysis in section 5, theorem 5.1 and expresion (47), in table 2, the numerical rate of

Table 1.Theoretical rate of order of convergence

| h | N | maximum absolute error | theoretical rate of convergence $P_N$ |
|---|---|---|---|
| $2^{-2}$ | 4 | 1.524973E-01 | 0.924164159 |
| $2^{-3}$ | 8 | 8.025333E-02 | 0.960505997 |
| $2^{-4}$ | 16 | 4.129170E-02 | 0.979805347 |
| $2^{-5}$ | 32 | 2.093688E-02 | 0.980787025 |
| $2^{-6}$ | 64 | 1.054281E-02 | 1.000856566 |
| $2^{-7}$ | 128 | 5.290204E-03 | 0.997424638 |
| $2^{-8}$ | 256 | 2.649828E-03 | average rate $P = \frac{1}{6}\sum P_N = 0.976708865$ |

order of onvergence and average rate of order of convergence are tabulated, considering the exact solution is not known (exact solution is not used for absolute error estimation). Problem (3) with $\lambda = 1$ is used for tabulation.

It is found that the numerical rate of order of convergence and average numerical rate of order of convergence are of one. From tables 1 and 2, it is observed that using problem (3) with $\lambda = 1$, we are able to get both theoretical and numerical rates of order of convergence as one. Similarly, both theoretical and numerical average rates of order of convergence are of one.

Table 2.Numerical rate of order of convergence

| h | N | maximum absolute error | numerical rate of convergence $P_N$ |
|---|---|---|---|
| $2^{-2}$ | 4 | 1.442880E-01 | 0.885127282 |
| $2^{-3}$ | 8 | 7.812326E-02 | 0.940381334 |
| $2^{-4}$ | 16 | 4.070965E-02 | 0.969606669 |
| $2^{-5}$ | 32 | 2.078819E-02 | 0.984659146 |
| $2^{-6}$ | 64 | 1.050521E-02 | 0.992289707 |
| $2^{-7}$ | 128 | 5.280732E-03 | 0.996135047 |
| $2^{-8}$ | 256 | 2.647459E-03 | average rate $P = \frac{1}{6}\sum P_N = 0.96136654$ |

**Experimental results with respect to error:** From the derivation of theorem 4.1 of section 4, it is observed that, the local and global truncation errors get increased from time steps $t = t_0 = a$ to $t = t_1$ and upto $t = t_N = b$; since the numerical method (2) is a recurrence relation between two consecutive time steps $t = t_i$ and $t = t_{i+1}$, the error in time step $t = t_i$ gets added with the error in the step $t = t_{i+1}$. Similarly the absolute and relative errors get increased from time steps $t = t_0 = a$ to $t = t_1$ and upto $t = t_N = b$. This observation is illustrated in table 3 and in figures 8 to 12.

In figure 8, curves with + sign, . sign and * sign represent exact solution, numerical solution and the absolute error respectively. It is observed from figure 8 that, as the absolute error gets deviated away upwards from the t-axis, the numerical solution gets deviated downwards away from the exact solution. If the absolute error is very closer to the t-axis as time is increased, then the numerical solution comes closer to the exact solution.

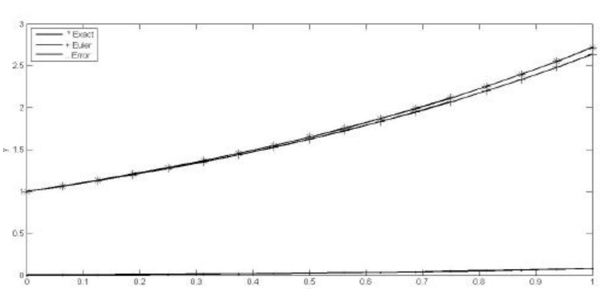

Figure 8.Exact solution, numerical solution and absolute error

In figure 9, curves with * sign and + sign represent absolute error and cummulative absolute error respectively. It is observed from figure 9 that, as the absolute error gets deviated away upwards from the t-axis, the

Table 3.Exact and numerical solutions, absolute and relative errors

| $t_i = h.i$ | $y(t_i)$ | $y_i$ | $y(t_i)$ - $y_i$ | cummulative | $1 - \frac{y_i}{y(t_i)}$ |
|---|---|---|---|---|---|
| $2^{-4}.0$ | 1.0000E+00 | 1.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 |
| $2^{-4}.1$ | 1.0645E+00 | 1.0625E+00 | 1.9945E-03 | 1.9945E-03 | 1.9131E-03 |
| $2^{-4}.2$ | 1.1331E+00 | 1.1289E+00 | 4.2422E-03 | 6.2367E-03 | 3.5907E-03 |
| $2^{-4}.3$ | 1.2062E+00 | 1.1289E+00 | 6.7674E-03 | 1.3004E-02 | 5.0545E-03 |
| $2^{-4}.4$ | 1.2840E+00 | 1.2744E+00 | 9.5961E-03 | 2.2600E-02 | 6.3246E-03 |
| $2^{-4}.5$ | 1.3668E+00 | 1.3541E+00 | 1.2757E-02 | 3.5357E-02 | 7.4192E-03 |
| $2^{-4}.6$ | 1.4550E+00 | 1.4387E+00 | 1.6280E-02 | 5.1637E-02 | 8.3551E-03 |
| $2^{-4}.7$ | 1.5488E+00 | 1.5286E+00 | 2.0200E-02 | 7.1837E-02 | 9.1478E-03 |
| $2^{-4}.8$ | 1.6487E+00 | 1.6242E+00 | 2.4551E-02 | 9.6388E-02 | 9.8112E-03 |
| $2^{-4}.9$ | 1.7551E+00 | 1.7257E+00 | 2.9374E-02 | 1.2576E-01 | 1.0358E-02 |
| $2^{-4}.10$ | 1.8682E+00 | 1.8335E+00 | 3.4710E-02 | 1.2576E-01 | 1.0801E-02 |
| $2^{-4}.11$ | 1.9887E+00 | 1.9481E+00 | 4.0606E-02 | 2.0108E-01 | 1.1150E-02 |
| $2^{-4}.12$ | 2.1170E+00 | 2.0699E+00 | 4.7110E-02 | 2.4819E-01 | 1.1415E-02 |
| $2^{-4}.13$ | 2.2535E+00 | 2.1993E+00 | 5.4277E-02 | 3.0246E-01 | 1.1605E-02 |
| $2^{-4}.14$ | 2.3989E+00 | 2.3367E+00 | 6.2164E-02 | 3.6463E-01 | 1.1729E-02 |
| $2^{-4}.15$ | 2.5536E+00 | 2.4828E+00 | 7.0833E-02 | 4.3546E-01 | 1.1793E-02 |
| $2^{-4}.16$ | 2.7183E+00 | 2.6379E+00 | 8.0353E-02 | 5.1581E-01 | 1.1805E-02 |

cummulative error gets deviated upwards away from the absolute error. If the absolute error is very closer to the t-axis as time increases, then the cummulative absolute error comes closer to the absolute error.

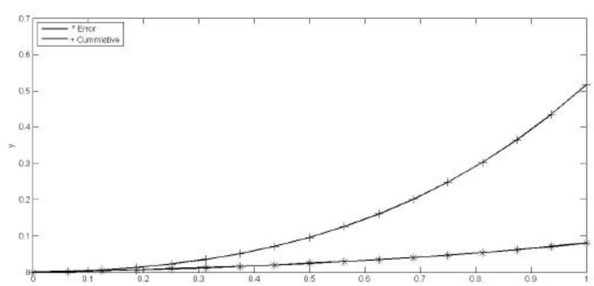


Figure 9.Absolute and cummulative absolute error

In figure 10, curves with * sign and + sign represent global truncation error and truncation error respectively. Figure 10 shows that, as the truncation error gets deviated away upwards from the t-axis, the global truncation error is deviated upwards away from the absolute error. If the truncation error is very closer to the t-axis as time increases, then the global truncation error comes closer to the truncation error.

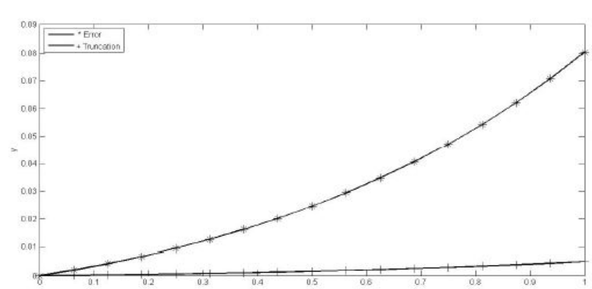


Figure 10.Truncation error and global truncaton error

In figure 11, curves with * sign and + sign represent relative error and cummulative relative error respectively. It is observed from figure 11 that, as the relative error gets deviated away upwards from the t-axis, the cummulative relative error gets deviated upwards away from the relative error. If the relative error is very closer or parallel to the t-axis as time increases, then the cummulative relative error comes closer to the relative error.

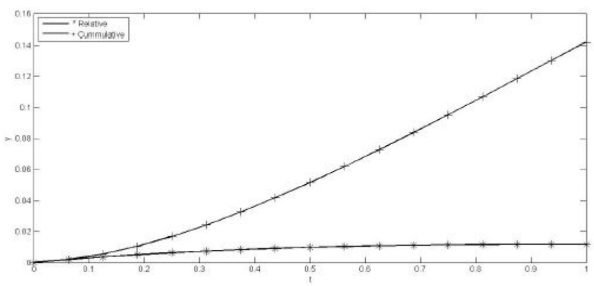

Figure 11.Relative error and cummulative relative error

## 7. CONCLUSION

The trajectory path of an aircraft from ground to space is modeled mathematically to locate the position of an aircraft at any time. Calculating the shape of the unknown curve which starts at a given point and satisfying the given differential equation is our problem. When the exact solution is not known, we have to go for a numerical method to find the solution numerically. Therefore, the numerical absolute error estimate derived and the numerical rate of order of convergence of the numerical solution obtained in section 5 are most suitable in practice.

When the exact solution is known, to test validity of the method, one can make use of the theoretical absolute error estimate derived and the theoretical rate of order of convergence of the numerical solution obtained in section 4. Electrical engineers and scientists engaged in interdisciplinary works can also follow this paper in the aspects of stability result, stability region in complex plane and theoretical and numerical rate of order of convergence based on the numerical method.

## REFERENCES


[1] K.Atkinson Kendall, An Introduction to Numerical Analysis, John Wiley and Sons, Delhi, 2008, pp. 1-648.

[2] K.Selvakumar, Uniformly Convergent Difference for Differential Equations with a Parameter, Ph.D. Thesis, Bharathidasan University, 1992, India.

[3] K.Selvakumar, Optimal Uniform Finite Difference Schemes of Order Two for Stiff Initial Value Problems, Communications in Numerical Methods in Engineering, Vol. 10, 1994, pp. 611-622.

[4] K.Selvakumar, Optimal Uniform Finite Difference Schemes of Order One for Singularly Perturbed Riccati Equation, Communicatioins in Numerical Methods in Engineering, Vol. 13, 1997, pp. 1-12.

[5] K.Selvakumar, A Computational Procedure for Solving a Chemical Flow-Reactor Problem Using Shooting Method, Applied Mathematics and Computation, Vol. 68, 1995, pp. 27-40.

[6] K.Selvakumar, A Computational Method for Solving Singularly Perturbation Problems Using Exponentially Fitted Finite Difference Schemes, Applied Mathematics and Computation, Vol. 66, 1994, pp. 277-292.

[7] K.Selvakumar and N.Ramanujam, Uniform Finite Difference Schemes for Singular Perturbation Problems Arising in Gas Porous Electrodes Theory, Indian Journal of Pure and Applied Mathematics, 1996, 293-305.

[8] K.Selvakumar, Uniformly Convergent Finite Difference Scheme for Singular Perturbation Problem Arising in Chemical Reactor Theory, International Journal of Computational Science and Mathematics, Vol. 2, 2010, pp. 77-90.

[9] K.Selvakumar, A Computational Method for Solving Singularly Perturbed Two Point Boundary Value Problems Without First Derivative Term, International e-Journal of Mathematics and Engineering, Vol. 70, 2010, pp. 694-707.

[10] K.Selvakumar, An Exponentially Fitted Finite Difference Scheme for Heat Equation, International e-Journal of Mathematics and Engineering, Vol. 79, 2010, pp. 776-797.

[11] K.Selvakumar, Initial Value Method for Solving Second Order Singularly Perturbed Two, Point Boundary Value Problem, International e-Journal of Mathematics and Engineering, Vol. 99, 2010, pp. 920-931.

[12] K.Selvakumar, Computational Procedure for Solving Singular Perturbation Problem Arising in Control System Using Shooting Method, International Journal of Computational Science and Mathematics, Vol. 1, No. 1, 2011, pp. 1-10.

[13] K.Selvakumar, Optimal and Uniform Finite Difference Scheme for Singularly Perturbed Riccati Equation, International Journal of Computational Science and Mathematics, Vol. 3, No. 1, 2011, pp. 11-18.

[14] K.Selvakumar, A Computational Method for Solving Singular Perturbation Problems Without First Derivative Term, International Journal of Computational Science and Mathematics, Vol. 3, No. 1, 2011, pp. 19-34.

[15] K.Selvakumar, A Computational Method for Solving Singularly Perturbed Initial Value Problems, International eJournal of Mathematics & Engineering, Vol. 161, 2012, pp. 487-1501.

[16] K.Selvakumar, Optimal and Uniform Finite Difference Scheme for Singularly Perturbed Riccati Equation, International Journal of Computational Science and Mathematics, Vol. 3, No. 1, 2011, pp. 11-18.

[17] K. Selvakumar, A Finite Difference Method for the Numerical Solution of First Order Nonlinear Differential Equation, International e-Journal Mathematics & Engineering, Vol, 84, 2012, pp. 1702-1709.

[18] K.Selvakumar, Explicit and not Fully Implicit Optimal and Uniform Finite Difference Schemes of Order One for Stiff Initial Value Problems, International Journal of Mathematical Science,Technology & Humanities, Vol. 53, 2012, pp. 561-576.

[19] K.Selvakumar, Application of Euler Method to Singular Perturbation Problems, DJ Journal of Engineering and Applied Mathematics, Vol. 4, No. 1, 2018, pp. 1-19, https://dx.doi.org/10.18831/djmaths.org/2018011001.

[20] J.C.Butcher, Numerical Methods for Ordinary Differential Equations in the $20^{th}$, Century Journal of Computational and Applied Mathematics, Vol. 125, No. 1-2, 2000, pp. 1-29, https://dx.doi.org/10.1016/S0377-0427(00)00455-6.

[21] C.W. Gear, Numerical Solutions of Ordinary Differential Equations: Is There Anything Left to do?, SIAM Review, Vol. 23, No. 1, 1981, pp. 10-24, https://dx.doi.org/10.1137/1023002.

[22] H.Lomas, Stable Implicit and Explicit Numerical Methods for Integratinng Quasi-Linear Differential Equations with Parastic Stiff and Parasitic Saddle Eigen Values, NASA Technical Note, 1968, pp. 1-30.

[23] D.Araceli Queiruga, E.Ascension Hernandez, V.Jesus Martin, R.Angel Martin del, B.P.Juan Jose and R.S.Gerardo, How Engineers Deal with Mathematics solving Differential Equation, Procedia Computer Science, Vol. 51, 2015, pp. 1977 - 1985, https://dx.doi.org/10.1016/j.procs.2015.05.462.

[24] R.Anthony and R.Philip, A first course in Numerical Analysis, Dover Publications, New York, 2001, pp. 1-542.