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RESEARCH ARTICLE

The Asymptotic Behavior of the Titchmarsh-Weyl m-function for a Dirac System on the Line

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ABSTRACT

The Titchmarsh-Weyl matrix m-function for the Dirac equation $y' = \begin{bmatrix} p & (\lambda + c + v_1) \\ (-\lambda + c + v_2) & -p \end{bmatrix} y$ on the line is shown through analysis of the Jost functions, to be asymptotically diagonal and $\lambda-$ independent as $|\lambda| \to \infty$ through complex values. An immediate consequence of this behavior is that the spectral function ρ associated with the equation on the line has an asymptotically diagonal linear form, and hence the absence of large eigen values for the associated operator is confirmed.
Keywords: Jost function, Dirac equation, Spectral function, Diagonal linear form, Eigen values.

1. INTRODUCTION

This paper deals with a one-dimensional Dirac system on the line namely,

$$y' = \begin{bmatrix} p & (\lambda + c + v_1) \\ (-\lambda + c + v_2) & -p \end{bmatrix} \tag{1}$$

where $y(x,\lambda) = \begin{bmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{bmatrix}$, $x \in i$, λ is a complex spectral parameter and c is a positive constant. Our main hypothesis is that the real-valued, Lebesgue measurable functions p , v_1 and v_2 verify the scattering condition.

$$\int_i (1+|x|)(|p(x)| + |v_1(x)| + |v_2(x)|)dx < \infty \tag{2}$$

The system (1) has been widely studied, with specific potential functions p, v_1 and v_2, in connection with physical phenomena. We mention here its association with a Bose gas [8] and monochromatic waves [9, 10], in which cases the condition (2) is a natural requirement. Subject to condition (2), equation (1) is known to be of limit-point type at both $x = \pm\infty$. By choosing a suitable domain (see [4], for example) one associates with (1) a self–adjoint operator formally defined by

$$L[y] = Jy' + [A+V]y \tag{3}$$

where J=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, A=$\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$ and V=$\begin{pmatrix} -v_2 & -p \\ -p & -v_1 \end{pmatrix}$.

The spectrum of L has been widely investigated ([1]-[5] for example) and is known to consist of a continuously differentiable essential spectrum $(-\infty, -c) \cup (c, \infty)$ and a finite number of eigenvalues in (-c, c). The study of the spectrum of L can be to a great extent facilitated by a matrix-valued function, $M(\lambda)$, which first introduced by Titchmarsh [7] for a particular case of (1). Besides "picking out" a basis for L $L^2(-infty, infty)$ solutions of problems associated with (1), this so-called Titchmarsh-Weyl m-function characterizes the spectrum and the resolvent set of L [4]. It has also been used as a tool in perturbation theory in connection with spectral concentration [5]. An analogue of the celebrated Levinson theorem [6] can also be derived from the behavior of $M(\lambda)$ as $\lambda \to \pm c$ [2].

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This note addresses the large λ behavior of $M(\lambda)$. Specifically, we show that the half-line m-functions associated with equation have conjugate behaviors as $|\lambda| \to \infty$ in the complex plain, resulting in an asymptotically diagonal matrix m-function for $(-\infty, \infty)$. Our method is based on the analysis of the so-called Jost functions associated with (1). In section 2, we define the quantities pertinent to our analysis and state their relevant properties. Some preliminary lemmas are also provided [2]. Section 3 brings the statement and proof of our main results.

2. PRELIMINARIES

The m-function associated with equation (1) on the whole line is defined in the following manner. Let the fundamental matrix solution for (1) be denoted by $Y(x,\lambda) = \begin{pmatrix} \theta_1(x,\lambda) \\ \theta_2(x,\lambda) \end{pmatrix}$, i.e. $Y(0,\lambda) = I_2$ for all λ. Then the half-line m-coefficients, $m\pm$ for system (1.1) at $c = \pm\infty$ are respectively defined by,

$$m\pm = -\lim_{x\to} \frac{\theta_1(x,y)}{\phi_1(x,y)}, \quad \Im(\lambda \neq 0) \tag{4}$$

Their analyticity properties and connection to the spectrum of operators induced by (1) on $(-infty, 0)$ and $(0, infty)$ are well known. In studying the non-homogeneous system for f in an appropriate domain, one naturally studies the homogeneous system (1).

$$y' = \begin{bmatrix} p & \lambda + c + v_1 \\ -\lambda + c + v_2 & -p \end{bmatrix} \tag{5}$$

Let us note that we define the operator L by the formal expression.

$$L[y] = Jy' + [A+V]y \tag{6}$$

where J=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, A=$\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$ and V=$\begin{pmatrix} -v_2 & -p \\ -p & -v_1 \end{pmatrix}$.

Then (1) and (5) are respectively equivalent to

$$L[y] = \lambda I y and L[y] = \lambda I y + f(x)$$

Now, the Green's function for (5) is given by,

$$G(x,t,\lambda) = \phi_+(x,\lambda)\left[m_-(\lambda) - m_+(\lambda)\right]^{-1} \phi_-^*(t,\bar{\lambda}), x > t$$

$$\phi_-(x,\lambda)\left[m_-(\lambda) - m_+(\lambda)\right]^{-1} \phi_-^*(t,\bar{\lambda}), x < t$$

where $\phi_+(x,\lambda)$ and $\phi_-(x,\lambda)$ are the unique (up to the constant multiples) solutions of (1) in $L^2(0,\infty)$ and $L^2(-\infty,0)$, respectively given by, $\phi_\pm(x,\lambda) = \theta(0,\lambda) + m_\pm(\lambda)\phi(x,\lambda)$, where $*$ denotes complex conjugate transpose and $\theta(x,\lambda)$ and $\phi(x,\lambda)$ are the solutions used in definition (4). It is easily seen that we may re-write $G(x,t,\lambda)$ as,

$$G(x,t,\lambda) = \begin{cases} Y(x,\lambda)M_1(\lambda)Y^*(t,\bar{\lambda}), & x > t \\ Y(x,\lambda)M_2(\lambda)Y^*(t,\bar{\lambda}), & x < t \end{cases} \tag{7}$$

where $Y(x,\lambda)$ is as above and $M_1 = \frac{1}{[m_- - m_+]}\begin{pmatrix} 1 & m \\ m_+ & m_+m_- \end{pmatrix}$, $M_2 = \frac{1}{[m_- - m_+]}\begin{pmatrix} 1 & m_+ \\ m_- & m_-m_+ \end{pmatrix}$

With λ-dependences suppressed, the M-function associated with (1) is then defined as,

$$M(\lambda) = \frac{1}{2}[M_1(\lambda) + M_2(\lambda)] \tag{8}$$

Its analyticity properties and relation to the spectrum of L are given in [1] and the references therein. We do, however, point out that M picks out L^2_{IR} solutions of (5) in the sense that,

$$y(x) = \int_{IR} G(x,t,\lambda) f(t) dt$$

which is the only L^2_{IR} solution of (5). The connection between $M(\lambda)$ and the spectral formula ρ of L is given by the Titchmarsh-Kodaira formula;

$$\rho(\mu_2) - \rho(\mu_1) = \frac{1}{\Phi} \lim_{\varepsilon \to 0^+} \int_{\mu_1}^{\mu_2} Im\, M(+i\varepsilon) d\mu \tag{9}$$

At points of continuity, μ_1, μ_2 of ρ. Let us now turn our attention to the so-called Jost functions, which is described as follows. Let $S = \lambda : Im\lambda \geqslant 0, \lambda \neq \pm c$. For $\lambda \varepsilon S$, set w= $[\lambda^2 - c^2]^{\frac{1}{2}}$, where the principal branch of $\sqrt{}$ is used, except on $(-\infty, -c)$, where it is defined to be continuous on S. If we let L denote the unperturbed or free operator, i.e., L at $V \equiv 0$, we see that the fundamental matrix for the free problem is given by,

$$E(x,\lambda) = \begin{pmatrix} coswx & \frac{\lambda+c}{w} \\ \frac{-w}{\lambda+c} sinwx & coswx \end{pmatrix},$$

Thus the variation of constants formula gives us the solution of L [y] = y as,

$$y(x) = E(x,\lambda)y(0) + \int_0^x E(x-t,\lambda)Q[y(t)]dt \tag{10}$$

where Q[y(t)] = JVy(t), and we have suppressed the λ-dependence. For any solution y, for i.e., given any y(0) in (10), of (1), the Jost function $A_y(\lambda)$ and $B_y(\lambda)$ are defined for $\lambda \in S$ by,

$$A_y(\lambda) = \left(\frac{w}{i(\lambda+c)}, 1 \right) y(0) + \int_0^\infty e^{iwt} \left(\frac{w}{i(\lambda+c)}, 1 \right) Q[y(t)]dt \tag{11}$$

$$B_y(\lambda) = \left(\frac{w}{i(\lambda+c)}, -1 \right) y(0) + \int_{-\infty}^0 e^{iwt} \left(\frac{w}{i(\lambda+c)}, -1 \right) Q[y(t)]dt \tag{12}$$

The next lemma established in [2] ensures that the Jost functions are analytic on Im $\lambda \neq 0$, bounded on $|\lambda|$>c, and have continuous extensions to $IR \backslash \{-c,c\}$; and thus can be analytically continued into Im $\lambda < 0$ by Schwartz reflection to obtain the following relation.

Lemma 2.1 [Lemma 4.1 of [1]]

$$\lim_{x\to\infty} e^{iwx} y(x) = A_y(\lambda) \begin{pmatrix} \frac{i(\lambda+c)}{2w} \\ \frac{1}{2} \end{pmatrix}, IMW > 0;$$

$$\lim_{x\to\infty} e^{-iwx} y(x) = B_y(\lambda) \begin{pmatrix} \frac{i(\lambda+c)}{2w} \\ -\frac{1}{2} \end{pmatrix}, IMW > 0;$$

We also have that $A_y(\lambda)$ and $B_y(\lambda)$ respectively determine the asymptotic phase of $y(x,\lambda)$ as $x \to +\infty$ and as $x \to -\infty$. This statement is stated below.

Lemma 2.2 [Lemma [3.7 of [2]]

Let $\lambda \to IR \backslash \{-c,c\}$. Then

$$y(x) = Re\{e^{-iwx} A_y(\lambda) \begin{pmatrix} \frac{i(\lambda+c)}{2w} \\ 1 \end{pmatrix}\} + o(1), as \;\; x \to \infty$$

$$y(x) = Re\{e^{iwx} A_y(\lambda) \begin{pmatrix} \frac{i(\lambda+c)}{2w} \\ 1 \end{pmatrix}\} + o(1), as \;\; x \to -\infty$$

The definition of the M-function features two solutions $\theta(x,\lambda)$ and $\phi(x,\lambda)$ of (1.1). We thus, as our project is to establish asymptotic behaviour of $M(\lambda)$ as $|\lambda| \to \infty$, turn to the behaviour of the Jost functions $A_\theta, A_\phi, B_\theta, B_\phi$ as $|\lambda \to \infty|$. These behaviours are established in [2] and are as follows:

Lemma 2.3 [Lemma 4.2 of [1]]

Let $\lambda \in S$ and set $v = \frac{1}{2}(v_1 + v_2)$. Then as $|\lambda| \to \infty$;

$$a) \quad A_\theta(\lambda) = -iexpa\left(-i\int_0^\infty v(t)dt\right) + o(1);$$

$$b) \quad A_\phi(\lambda) = expa\left(-i\int_0^\infty v(t)dt\right) + o(1);$$

$$c) \quad B_\theta(\lambda) = -iexpa\left(-i\int_{-\infty}^0 v(t)dt\right) + o(1);$$

$$d) \quad B_\phi(\lambda) = -expa\left(-i\int_{-\infty}^0 v(t)dt\right) + o(1).$$

3. MAIN RESULTS

With these preliminaries, our main result may then be stated as follows:

Theorem 3.1

Let Im$\lambda > 0$. Then as $|\lambda| \to \infty$ we have,

$$M(\lambda) = \begin{pmatrix} \frac{1}{2}i + o(1) & o(1) \\ o(1) & \frac{1}{2}i + o(1) \end{pmatrix}$$

The corresponding behaviour for Im$\lambda < 0$ is

$$M(\lambda) = \begin{pmatrix} -\frac{1}{2}i + o(1) & o(1) \\ o(1) & -\frac{1}{2}i + o(1) \end{pmatrix}$$

Proof

The last statement follows as the first Lemma 2 and the well known properties.

$$m_\pm(\bar{\lambda}) = \overline{m_\pm(\lambda)}$$

To obtain the first statement, we note that the definition (4) and Lemma (2.1) yield,

$m_+ = -\frac{A_\theta}{A_\phi}$ and $m_- = -\frac{B_\theta}{B_\phi}$, we then see that definition (8) gives $m_{11} = \frac{A_\phi B_\phi}{F}$, $m_{22} = \frac{A_\theta B_\theta}{F}$, and $m_{12} = m_{21} = \frac{A_\theta B_\phi + A_\phi B_\theta}{2F}$.

Hence the Lemma completes the proof of the theorem. An immediate corollary of theorem 3.1, coupled with the formula (9) is the following:

Corollary 3.1

In condition (2), let L be the operator expressed in (9) and let $\rho(\lambda)$ denote its spectral function. Then as $|\lambda|$ in IR, $\frac{d\rho(\lambda)}{d\rho} = \frac{1}{2\rho}I_2 + o(1)$, where $o(1)$ means a matrix, in which all of the elements are $o(1)$.

4. CONCLUSION

1. The theorem shows that the matrix $M(\lambda)$ is not continuous across IR as $|\lambda| \to \infty$ in C. In particular, for $\lambda \in IR$ is large enough, it is an independent demonstration of the well-known fact that is in the continuous spectrum of L ; and also that $\{\lambda | Im\lambda \neq 0, |\lambda| > R$ for some R large enough$\}$ is in the resolvent set of L. We do, however, note that our theorem shows that the off-diagonal elements of M are asymptotically continuous across IR.

2. Corollary 3.1 shows that the spectral function of L is asymptotically diagonally linear in λ with constant off-diagonal terms, and in particular, L has no embedded eigenvalues.

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