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RESEARCH ARTICLE

Application of Euler Method to Singular Perturbation Problems

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ABSTRACT

Application of Euler's forward method for the numerical solution of singularly perturbed differential equation is presented. Stability region and order finger star are provided to show the order of convergence. The rate of theoretical and numerical order of convergence is derived. Experimental results are presented to show the performance of the method both numerically and graphically, based on the metrices such as amplification factor, stability function, stability region and theoretical and numerical rate of order of convergence.
Keywords: Singular perturbation problems, Euler's method, Stability region, Order star finger, Amplification factor.

1. INTRODUCTION

Consider the singularly perturbed initial value problem

$$\varepsilon y^{'}(t) = f(t,y(t)),\ t \in [a,b], y(a) = \phi \tag{1}$$

where ε is a small parameter such that $0 < \varepsilon << 1$. It is a problem of calculating the shape of the unknown curve which starts at a given point t=a and satisfies the given singularly perturbed differential equation. In [1,10], the notion of stiff and singular perturbation problems are presented.

In 1769-70, Atkinson [4], Leonhard Euler in his book, Institutionum Calculi Integralis, designed a numerical method for the solution of the problem (1) and it is of the form

$$y_{i+1} \quad = \quad y_i + \frac{h}{\varepsilon} f(t_i, y_i),\ i = 0,1,2,\cdots,N-1,\ y_0 = \phi \tag{2}$$

where $\{a = t_0, t_1, t_2, \cdots, t_N = b\}$ is a sequence of points in $[a,b]$, using the step size h, $t_i = t_0 + ih$ and $h = t_{i+1} - t_i$, $i = 0,1,2,\cdots,N-1$. The method (2) is consistent with the problem (1) as step size h approaches zero ($h \to 0$). The order of the method (2) is one for a fixed ε. This means,

(1) The error per step is proportional to the square of the step size.

(2) The error at a given time is proportional to the step size.

To design a numerical method, the deeper knowledge of the stability of the difference equation must be known to a designer. For a designer, some tips are given by [5-8], which examined the development of numerical solution techniques from the identification of the problem to the never-final preparation of automatic codes for the solution of classes of similar problems. It is a survey of recent works that have advanced the state of art or offers promise of the future. It discusses a number of problems that are only a part of the way along the path of development.

Euler's forward method serve as the basis to construct more complex numerical methods for singular perturbation problems. Complex numerical methods have been generated from Euler's forward method in the works of Selvakumar [10-27]. A program in Maple, is presented to solve initial value problems numerically by [9]. A program in Matlab, is presented to solve second order initial value problem numerically in [3].

For singular perturbation problems, in the literature, the theoretical rate of order of convergence is derived. The procedure adopted to find error estimate cannot be applicable to higher order methods. This motivates to derive theoretical rate of order of convergence and this procedure must be applicable to higher order methods. The numerical rate of order of convergence is not derived in the literature and it is derived in this paper. For the

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completeness of this paper both deivations are included. Finally, stability regions, order star, order star finger region, relative stability region, relative absolute region and order graph are not explained in the literature for singular perturbation problems. In this paper, we present all these concepts.

In brief, this paper presents a detailed analysis on Euler's forward method for the numerical solution of the first order singularly perturbed differential equations. In section 2, stability analysis is given and in section 3, stability functin, stability region, order star fingers for singularly perturbed problem are presented. Stability region analysis is given in section 4. Order star finger and region analysis is given in section 5. In section 6, the theoretical rate of order of convergence is derived. In section 7 , the numerical rate of order of convergence is derived. Finally, in section 8, experimental results are presented to show the performance of the method both numerically and graphically.

Throughout this paper C is a constant independent of i, h and ε.

2. STABILITY RESULT

In this section, the stability analysis of the singularly perturbed problem is given. On applying the method (2) to the problem [2],

$$\varepsilon y^{'}(t) = -y(t), y(0) = 1 \tag{3}$$

whose exact solution is $y(t) = exp(-\frac{t}{\varepsilon})$ We have the difference equation as

$$\frac{y_{i+1}}{y_i} = 1 - \frac{h}{\varepsilon} \tag{4}$$

The corresponding amplification factor $Q(\frac{h}{\varepsilon})$ is

$$Q(\frac{h}{\varepsilon}) = 1 - \frac{h}{\varepsilon} \tag{5}$$

Stability: A numerical method is stable if

$$\left| Q(\frac{h}{\varepsilon}) \right| \leq 1 \tag{6}$$

Using (6), one can determine the maximum step size h for which the method is stable. The equation (6) holds only when $0 \leq h \leq 2\varepsilon$.

3. STABILITY REGION

In this section, the stability region of the solution of the singularly perturbed problem is presented. The value of ε considered is 0.75 for Matlab contour.

Stability region of exact solution: The stability function of the exact solution of the problem (3) is given by

$$R(z) = exp(-\frac{z}{\varepsilon}) \tag{7}$$

where z is a complex number and the stability region is on right half of the complex plane which is presented in figure 1 as a blue colour shaded portion. For $\varepsilon = 1$ the stability region is the entire right half plane. So in the brown shaded region it is unstable when $\varepsilon = 0.75$

Stability region of numerical solution: On applying Euler's method (2) to the problem (3), the stability function of the numerical solution is given by

$$R(z) = 1 - \frac{z}{\varepsilon} \tag{8}$$

where z is a complex number and the stability region is a region inside circular region with center $(\varepsilon, 0)$ and radius ε on the right half complex plane which is presented in figure 2.

On applying the Euler forward method to the problem (3), the numerical solution is stable inside the contour and unstable if the values of z is outside the region $\{z \in C / |\varepsilon - z| \leq \varepsilon\}$

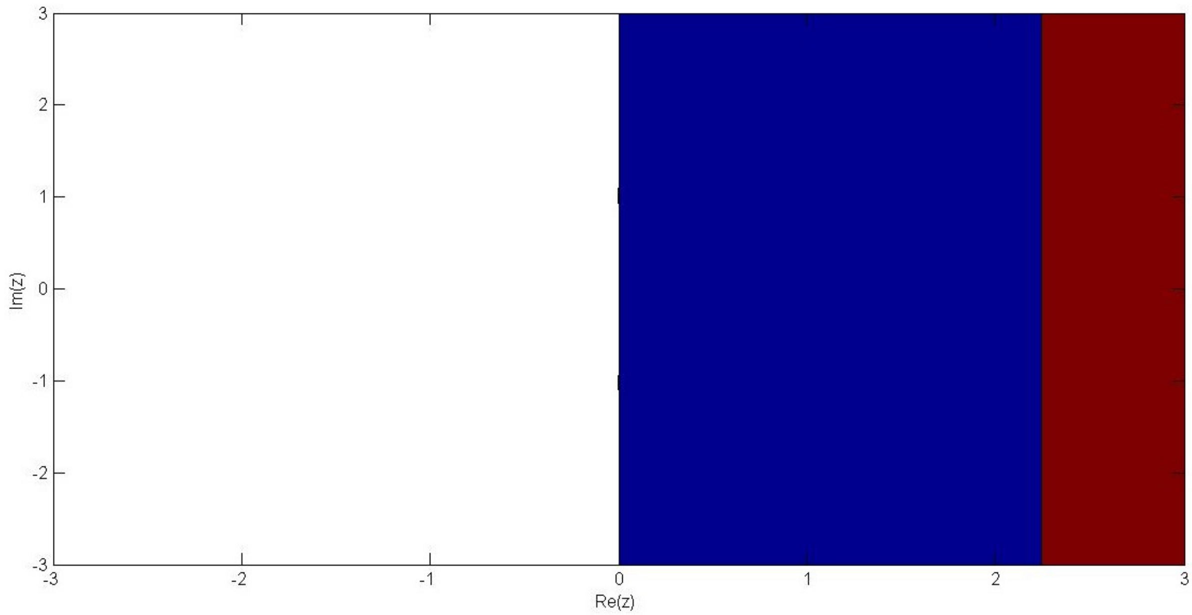


Figure 1.Stability region of exact solution of singular perturbation problem

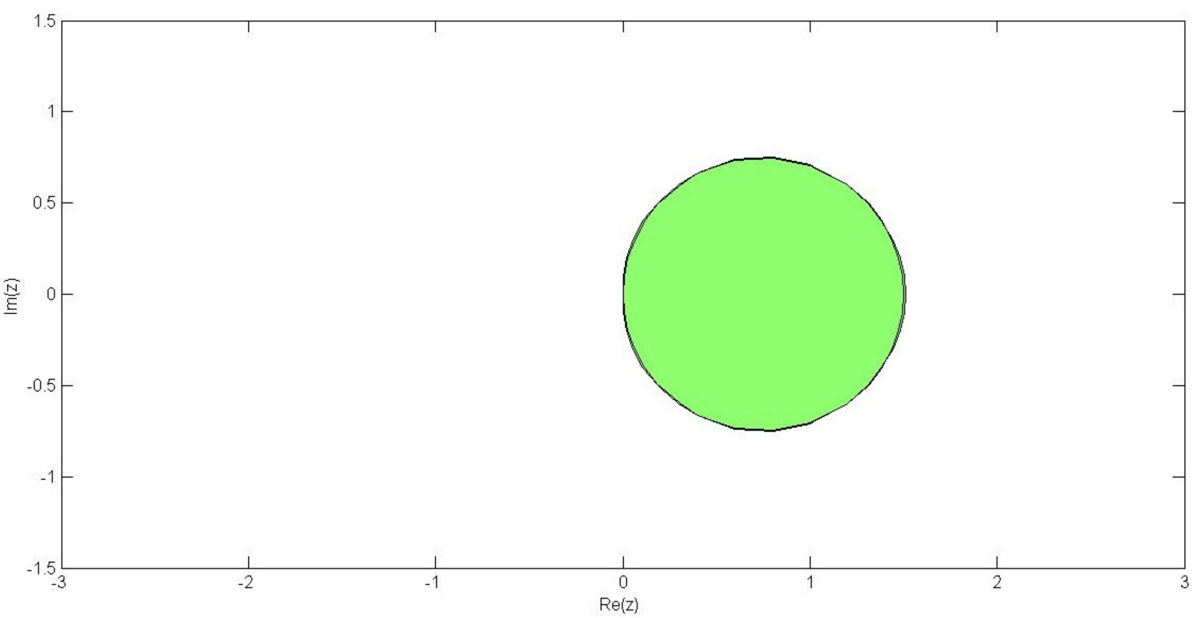


Figure 2.Stability region of numerical solution of the problem

Order star finger: The degree of the stability function of the singularly perturbation problem on applying Euler's method is one and so the order of the numerical method is one. In figure 3, only one finger lies inside the stability region and so the method is of order one.

Order star finger region of numerical method: The order star finger region of Euler's method (2) is given in figure 4. The region is in blue colour.

Relative stability region: The relative stability region or the order star of first kind for the Euler's forward method is given in figure 4. The region is in brown colour.

Absolute relative stability region of numerical method: Absolute stability region for the Euler forward method obtained on relative comparison with one is given in figure 5.

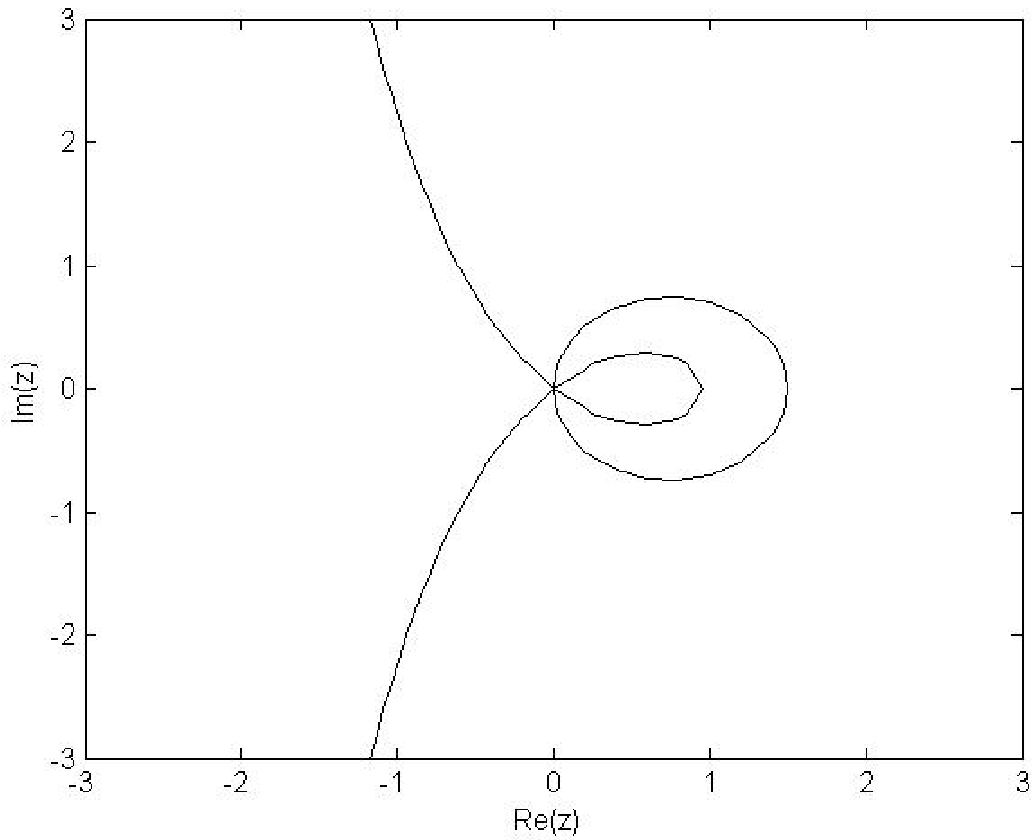


Figure 3.Order star-finger of the method for the problem

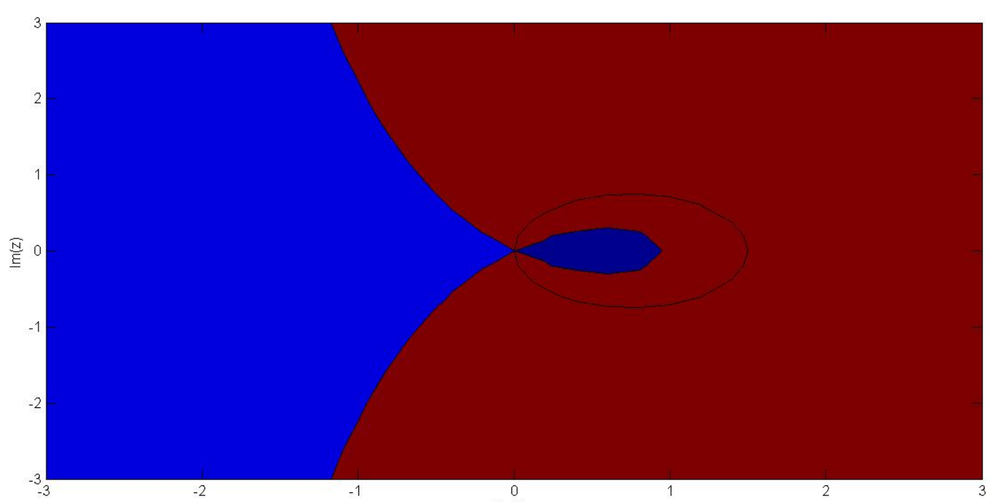


Figure 4.Order star-finger region of numerical method

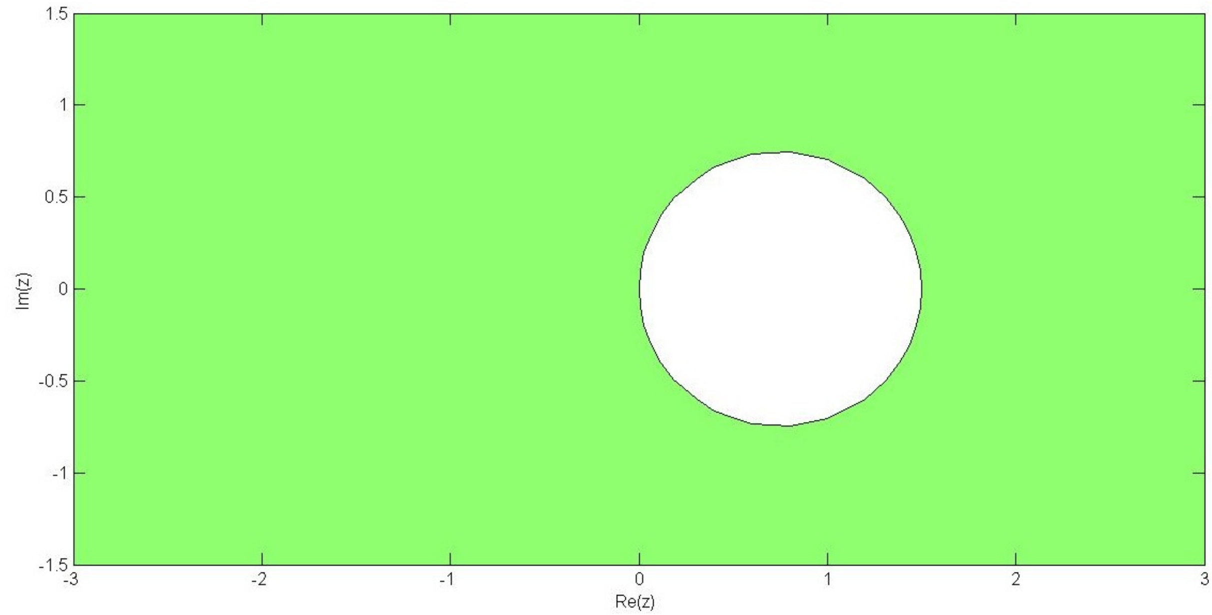


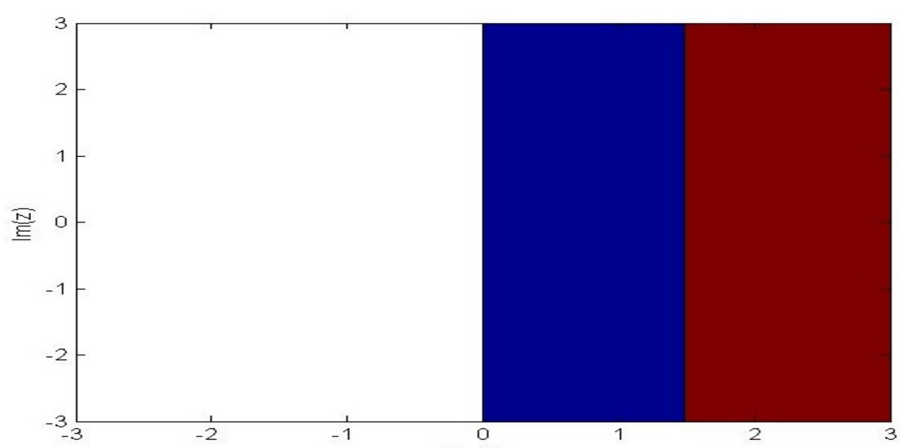
Figure 5.Absolute relative stability region of numerical method

From the above analysis, it is observed that the numerical method is of order one.

4. STABILITY REGION ANALYSIS

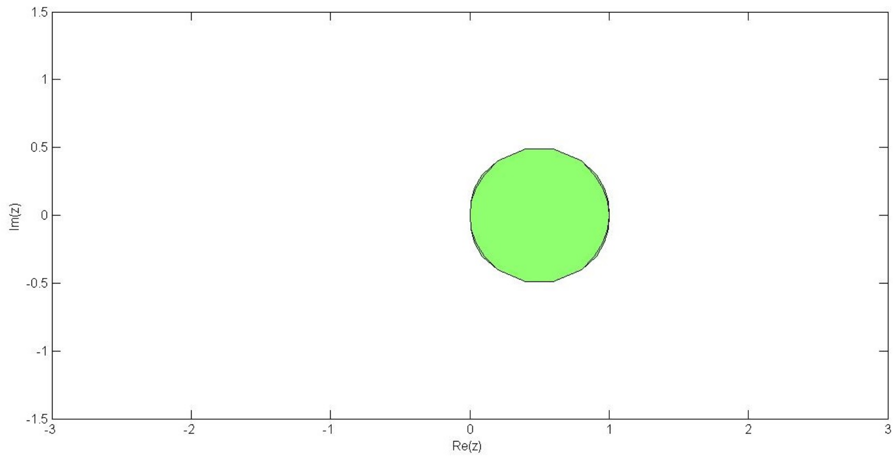
In this section, the stability regions with respect to the exact and numerical solutions are provided in the figures 6, 7 and 8 for $\varepsilon = 0.5$, 0.25 and 0.125 respectively. Region shaded by blue colour refers to stability region and brown colour region is the corresponding unstable region with respect to exact solution. The green colour region is the stable region and white colour region on the right half plane is the unstable region with respect to numerical solution.

It is observed that, on decreasing the value of ε, both stability region of exact solution and numerical solution get shrinked. The exact solution shrinks parallel to imaginary axis.



(a). Exact solution

The numerical region shrinks towards the origin since it is the region inside the contour with center $(\varepsilon, 0)$ and radius ε. This shows, no region for small ε which indicates the basic property of the singular perturbation problem. That is, as ε is very small ε approaches zero and the differential equation (1) reduces to the algebraic equation $f(t, y(t)) = 0, t \in [a, b]$ This algebraic equation can be solved for the unknown y(t).

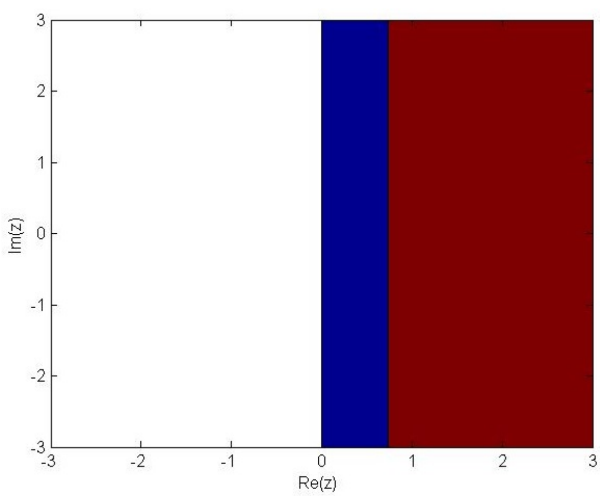


(b). Numerical solution

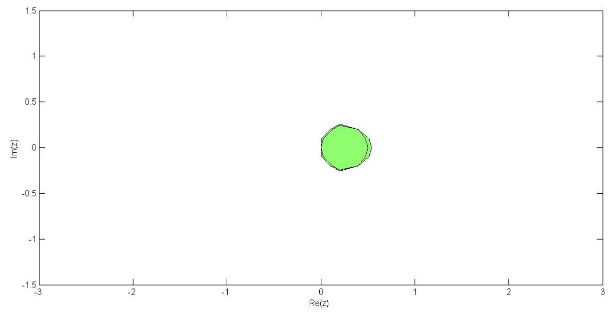
Figure 6.Stability regions of exact and numerical solutions, $\varepsilon = 0.5$

5. ORDER STAR FINGER AND REGIONS

In this section, the order star finger and order star finger regions with respect to the exact and numerical solutions are provided in the figures 9, 10, 11 and 12 for ε = 0.75, 0.5, 0.25 and 0.125 respectively. Blue and brown shaded regions refer to order star finger region and relative stability regions respectively. It is observed that there is only one finger inside the stability region and so the order of the method is one.

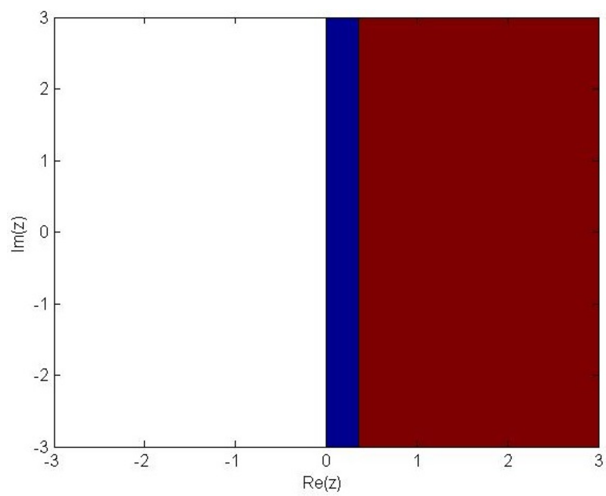


(a). Exact solution

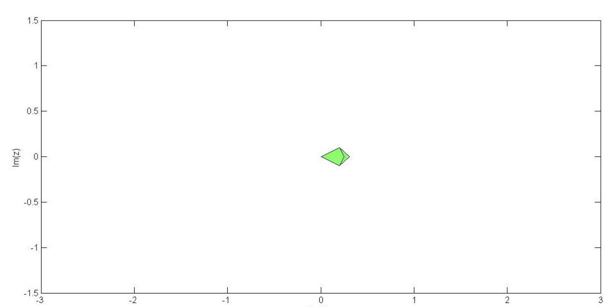


(b). Numerical solution

Figure 7.Stability regions of exact and numerical solutions, $\varepsilon = 0.25$



(a). Exact solution



(b). Numerical solution

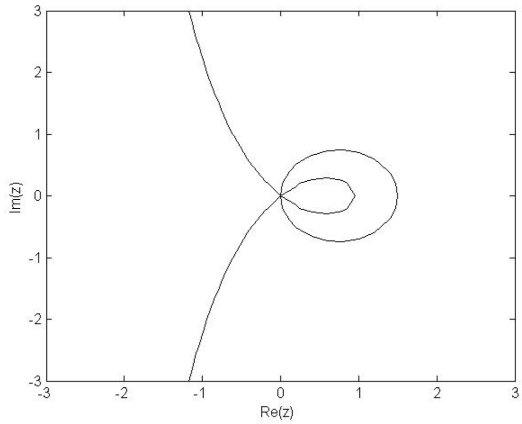
Figure 8.Stability regions of exact and numerical solutions, $\varepsilon = 0.125$

6. THEORETICAL RATE OF ORDER OF CONVERGENCE

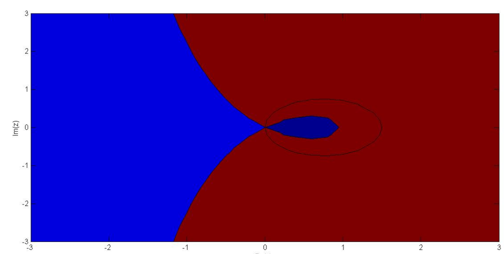
In this section, the theoretical rate of order of convergence of the numerical method is derived. The main result of this section is stated in the following theorem.

Theorem 6.1

If $y(t)$ is the solution of the given differential equation (1) and y_i is the numerical solution of the Euler's

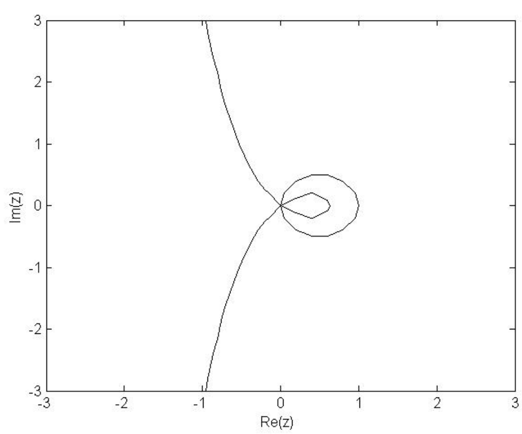


(a). Exact solution



(b). Numerical solution

Figure 9.Order star finger and regions of numerical solution, $\varepsilon = 0.75$



(a). Exact solution

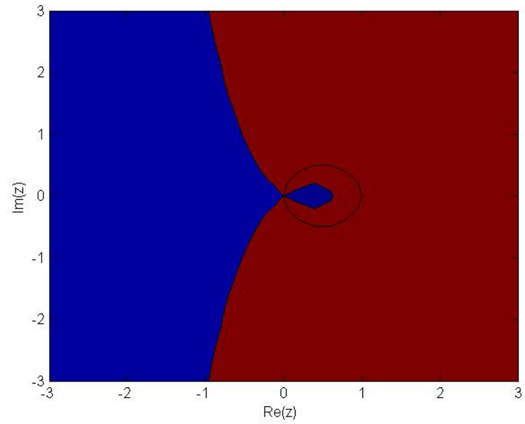
method (2), then, for i=0(1)N, we have an error estimate of the form,

$$|y(t_i) - y_i| \leq Ch^1 \tag{9}$$

where C is independent of i and h

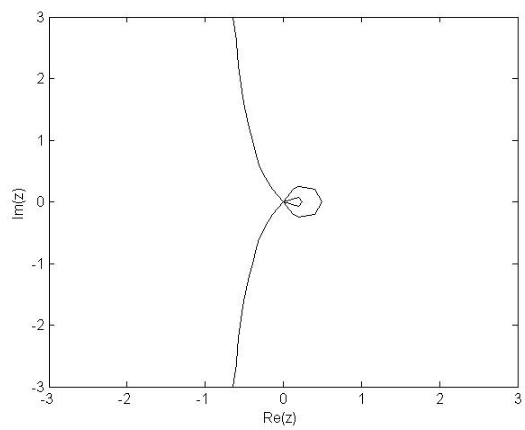
Proof

The solution y(t) of (1) at the nodal point $t = t_{i+1}$ can be expressed in terms of solution y(t) at the nodal

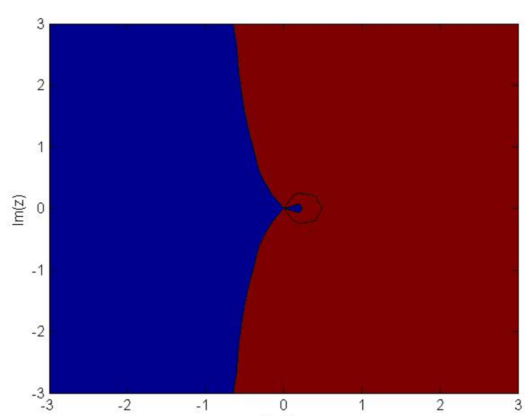


(b). Numerical solution

Figure 10.Order star finger and regions of numerical solution, $\varepsilon = 0.5$

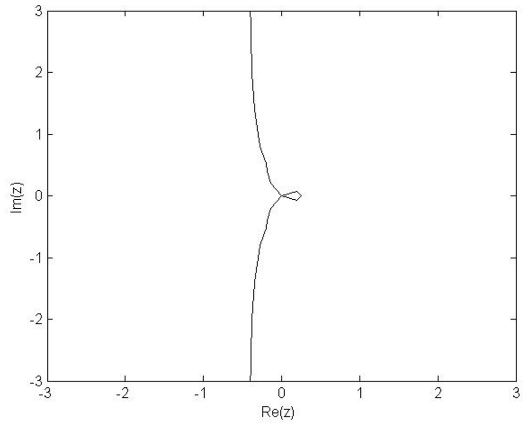


(a). Exact solution

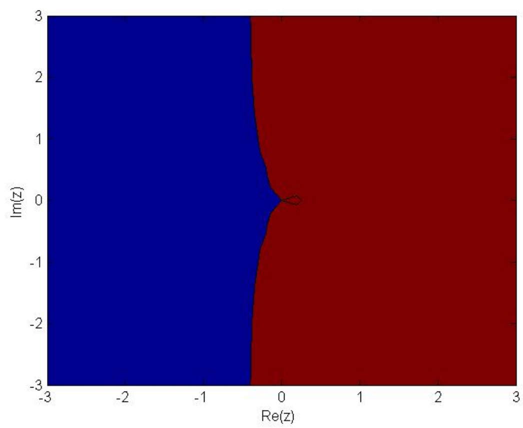


(b). Numerical solution

Figure 11.Order star finger and regions of numerical solution, $\varepsilon = 0.25$



(a). Exact solution



(b). Numerical solution

Figure 12.Order star finger and regions of numerical solution, $\varepsilon = 0.125$

point at $t = t_i$, using Taylor's series as follows:

$$y(t_{i+1}) = y(t_i) + hy^{'}(t_i) + \frac{h^2}{2!}y^{''}(t_i) + O(h^3),\ y(t_0) = \phi \tag{10}$$

where $h = t_{i+1} - t_i$, $i = 0, 1, \cdots N-1$

The numerical method (2) can be expressed as

$$y_{i+1} = y_i + \frac{h}{\varepsilon}f(t_i, y_i), y_0 = \phi \tag{11}$$

where $h = t_{i+1} - t_i$ and $y_i = y(t_i)$, $i = 0, 1, \cdots N$

Equation (11) can be rewritten as

$$y_{i+1} = y_i + \frac{h}{\varepsilon}y^{'}(t_i), y_0 = \phi \tag{12}$$

From equations (10) and (12), we have

$$\varepsilon[y(t_{i+1}) - y_{i+1}] = \varepsilon[y(t_i) - y_i] + \frac{\varepsilon h^2}{2}y^{''}(t_i) + O(\varepsilon h^3)$$

$$\varepsilon e_{i+1} = \varepsilon e_i + \frac{\varepsilon h^2}{2}y^{''}(t_i) + O(\varepsilon h^3), e_0 = 0 \tag{13}$$

where $e_i = y(t_i) - y_i, i = 0, 1, \cdots N$ The error per step, Local Truncation Error(LTE) is given by

$$LTE = \frac{\varepsilon h^2}{2} y''(t_i) + O(\varepsilon h^3) \quad (14)$$

The error at a given time t is termed as Global Truncation Error [GTE] and can be obtained from (15) by rewritting (15) as follows:

$$\varepsilon D_+ e_i = \frac{\varepsilon h^1}{2} y''(t_i) + O(\varepsilon h^2)$$

$$GTE = \frac{\varepsilon h^1}{2} y''(t_i) + O(\varepsilon h^2) \quad (15)$$

From equation (14), we have to find the error estimate for the absolute error. For i=0,1,2,...,N-1, we have

$$e_1 = e_0 + \frac{h^2}{2} y''(t_0) + O(h^3), e_0 = 0 \quad (16)$$

$$e_2 = e_1 + \frac{h^2}{2} y''(t_1) + O(h^3) \quad (17)$$

$$e_i = e_{i-1} + \frac{h^2}{2} y''(t_{i-1}) + O(h^3) \quad (18)$$

Continuing like this finally,

$$e_N = e_{N-1} + \frac{h^2}{2} y''(t_{N-1}) + O(h^3) \quad (19)$$

From equations (17) to (20), we have,

$$e_N = \frac{h^2}{2} \sum_{j=0}^{N-1} y''(t_j) + O(h^3) \quad (20)$$

Let k=max$\left| y''(t_j) \right. for j = 0(1)N-1$, then,

$$|e_N| \leq \frac{h^2}{2} \sum_{j=0}^{N-1} \left| y''(t_j) \right| + O(h^3) \quad (21)$$

$$|e_N| \leq \frac{h^2}{2} \sum_{j=0}^{N-1} k + O(h^3)$$

$$|e_N| \leq \frac{h^2}{2} Nk + O(h^3)$$

Since $N = \frac{b-a}{h}$, we have

$$|e_N| \leq Ch \quad (22)$$

Now for i=0 and i=N we have the estimate. And for i=i,2,...N-1 we shall find the error estimate by taking the equation (19)

$$e_i = \frac{h^2}{2} \sum_{m=0}^{i-1} y''(t_m) + O(h^3)$$

$$|e_i| \leq \frac{h^2}{2} \sum_{m=0}^{i-1} \left| y''(t_m) \right| + O(h^3)$$

Since

$$\sum_{m=0}^{i-1}\left|y^{''}(t_m)\right| \leq \sum_{j=0}^{N-1}\left|y^{''}(t_j)\right|$$

we have,

$$|e_i| \leq \frac{h^2}{2}\sum_{j=0}^{N-1}\left|y^{''}(t_j)\right| + O(h^3) \tag{23}$$

The right hand side of equation (22) is same as in equation (24), and so,

$$|e_i| \leq Ch \tag{24}$$

for i=1(1)N-1. Finally, from (17), (23) and (25) the desired estimate (9) follows for i=0(1)N

$$|e_i| \leq Ch \tag{25}$$

Hence the proof.

Theoretical rate of order of convergence If $y(t)$ is the solution of the given differential equation (1) and y_i is the numerical solution of the numerical method and, if we have an error estimate of the form,

$$max_{0\leq i\leq N}|y(t_i) - y_i| \leq Ch^n \tag{26}$$

then n is the theoretical rate of order of convergence of the method and it can be obtained from the estimate (27).

Rewrite (27) as

$e_i^h = Ch^n$ and $e_{2i}^{\frac{h}{2}} = C\frac{h^n}{2^n}$, then taking the ratio we get the theoretical rate of order of convergence of a numerical method as

$$n = \frac{log(\frac{e_i^h}{e_{2i}^{\frac{h}{2}}})}{log2} \tag{27}$$

where $e_i^h = y(t_i) - y_i^h$ and $e_{2i}^{\frac{h}{2}} = y(t_{2i}) - y_{2i}^{\frac{h}{2}}$. Here y_i^h stands for the numerical solution got by using step size h and $y_{2i}^{\frac{h}{2}}$ stands for the numerical solution got by using step size $\frac{h}{2}$. From (28) and (9) the theoretical rate of order of convergence of Euler's method is of one (n=1).

When exact solution is not known finding out the numerical rate of order of convergence is the next task. To solve this, an alternate numerical rate of order of convergence is derived in the next section.

7. NUMERICAL RATE OF ORDER OF CONVERGENCE

In this section, the numerical rate of order of convergence of the numerical method is derived. The main result of this section is stated in the following theorem.

Theorem 7.1

Let $y(t)$ be the solution of the given differential equation (1) and y_i^h and $y_{2i}^{\frac{h}{2}}$ be the numerical solution of the Euler's method (2) using step sizes h and $\frac{h}{2}$ respectively. Then, for i=0(1)N, we have an error estimate of the form,

$$\left|y(t_i) - y_i^h\right| = \left|2[y_{2i}^{\frac{h}{2}} - y_i^h]\right| \leq Ch^1 \tag{28}$$

where C is independent of i and h.

Proof

Let $w_i = 2[y_{2i}^{\frac{h}{2}} - y_i^h]$ for i=9(1)N. For $i = 0$,$w_0 = 0$, and for i=1(1)N we have, from (2),

$$w_{i+1} = w_i + \frac{h}{\varepsilon}[f(t_{2i}, y_{2i}^{\frac{h}{2}}) + f(t_{2i+1}, y_{2i+1}^{\frac{h}{2}})], w_0 = 0 \tag{29}$$

Using the procedure in theorem 6.1, equation (30) gets the form

$$\varepsilon w_{i+1} = \varepsilon w_i + \frac{\varepsilon h^2}{2}y^{''}(t_i) + O(\varepsilon h^3) \tag{30}$$

From (31), the error per step is given by

$$LTE = \frac{\varepsilon h^2}{2} y''(t_i) + O(\varepsilon h^3) \tag{31}$$

Equation (31) can be rewritten as,

$$\varepsilon D_+ w_i = \tfrac{\varepsilon h^1}{2} y''(t_i) + O(\varepsilon h^2)$$

and hence the error at a given time t is given by

$$GTE = \frac{\varepsilon h^1}{2} y''(t_i) + O(\varepsilon h^2) \tag{32}$$

Now, the equation (31) is for i=1(1)N. Adopting the procedure followed in theorem 6.1, for i=1(1)N, we have

$$w_i = w_{i-1} + \frac{h^2}{2} y''(t_{i-1}) + O(h^3) \tag{33}$$

Now, for i=1(1)N,we have the relation

$$w_i = \frac{h^2}{2} \sum_{m=0}^{i-1} y''(t_m) + O(h^3) \tag{34}$$

Following the procedure in theorem 6.1,

$$w_i \leq \frac{h^2}{2} \sum_{m=0}^{N-1} y''(t_m) + O(h^3) \tag{35}$$

Taking absolute value on both sides,

$$|w_i| \leq \frac{h^2}{2} \sum_{m=0}^{N-1} \left| y''(t_m) \right| + O(h^3) \tag{36}$$

Let $k = max \left| y''(t_m) \right|$, for m=0,1,2,...N-1, then, for i=1(1)N,

$$|w_i| \leq \frac{h^2}{2} \sum_{m=0}^{N-1} k + O(h^3) \leq \frac{h^2}{2} NK + O(h^3) \leq \frac{h^1}{2(b-a)} K + O(h^3) \tag{37}$$

From (30) and (38), we have the required result, for all i=0(1)N

$$|w_i| \leq Ch \tag{38}$$

Hence the proof.

Numerical rate of order of convergence: Let $y(t)$ be the solution of the given differential equation (1) and y_i^h and $y_{2i}^{\frac{h}{2}}$ are the numerical solution of the numerical method using step sizes h and $\frac{h}{2}$ respectively. Then, we have an error estimate of the form,

$$max_{0 \leq i \leq N} \left| y(t_i) - y_i^h \right| = max_{0 \leq i \leq N} \left| \frac{2^n}{2^n - 1} [y_{2i}^{\frac{h}{2}} - y_i^h] \right| \leq Ch^n \tag{39}$$

Here n is the numerical rate of order of convergence of the method and it can be obtained from the estimate (40). Rewrite (40) as $w_i^h = Ch^n$ and $w_{2i}^{\frac{h}{2}} = C\frac{h^n}{2^n}$, and then taking the ratio we get the numerical rate of order of convergence of a numerical method as

$$n = \frac{log(\frac{w_i^h}{w_{2i}^{\frac{h}{2}}})}{log 2} \tag{40}$$

where
$w_i = 2[y_{2i}^{\frac{h}{2}} - y_i^h]$ and $w_{2i} = 2[y_{4i}^{\frac{h}{4}} - y_{2i}^{\frac{h}{2}}]$

Here $y_{2i}^{\frac{h}{2}}$ stands for the numerical solution got by using step size $\frac{h}{2}$ and $y_{4i}^{\frac{h}{4}}$ stands for the numerical solution got by using step size $\frac{h}{4}$. From (40) and (41) the numerical rate of order of convergence of Euler's method is of one (n=1).

8. EXPERIMENTAL RESULTS

In this section, both numerical and graphical results for singular perturbation problems are given in the interval [0, 1]. Test problems considered are

Test problem 1: $\varepsilon y^{'} + y = 0, y(0) = 1$

Test problem 2: $\varepsilon y^{'} + y^2 = 0, y(0) = 1$

Theoretical rate of order of convergence: The theoretical rate of order of convergence of a numerical method P_N when the exact solution is known is defined as

$$P_N = \frac{log(\frac{e_i^h}{e_{2i}^{\frac{h}{2}}})}{log2}$$

where $e_i^h = y(t_i) - y_i^h$ and $e_{2i}^{\frac{h}{2}} = y(t_{2i}) - y_{2i}^{\frac{h}{2}}$

Here, N refers to number of nodal points on using a particular step size h. It is observed from table 1 that the theoretical rate of order of convergence is one, for the test problem 1.

Table 1.Theoretical rate of order of convergence, $\varepsilon = 0.125$

h	N	maximum absolute error	theoretical rate P_N
2^{-3}	8	6.302097E-01	-
2^{-4}	16	1.178794E-01	2.418520338
2^{-5}	32	5.147319E-02	1.118871455
2^{-6}	64	2.427053E-02	1.084615582
2^{-7}	128	1.180531E-02	1.039769687
2^{-8}	256	5.024152E-03	1.232483909
2^{-9}	512	2.892917E-03	0.796355188
2^{-10}	1024	1.441725E-03	1.004725924
2^{-11}	2048	1	average rate $P = \frac{1}{7}\sum P_N = 1.242191593$

Numerical rate of order of convergence: The numerical rate of order of convergence of a numerical method P_N when the exact solution is not known is defined as

$$P_N = \frac{log(\frac{w_i^h}{w_{2i}^{\frac{h}{2}}})}{log2}$$

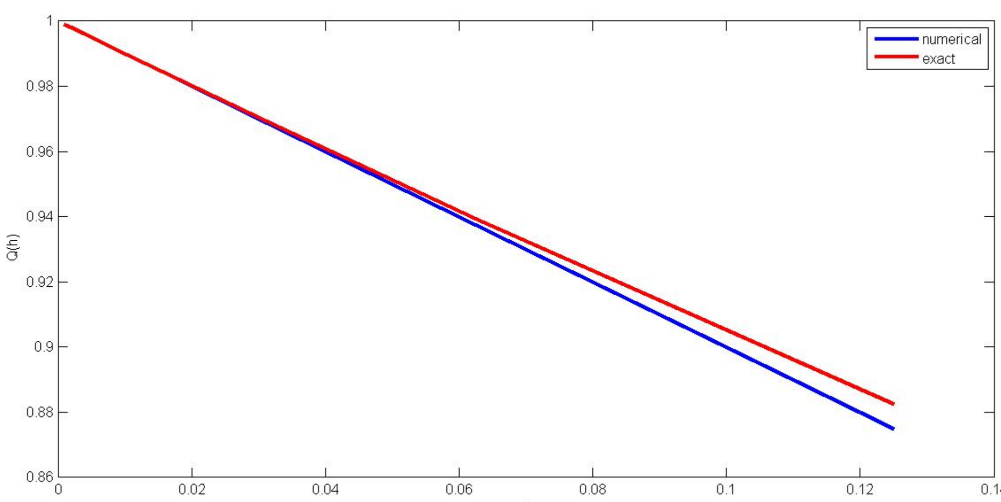
where $w_i = 2[y_{2i}^{\frac{h}{2}} - y_i^h]$ and $w_{2i} = 2[y_{4i}^{\frac{h}{4}} - y_{2i}^{\frac{h}{2}}]$. Here N refers to number of nodal points on using a particular step size h. It is observed from table 2 that the numerical rate of order of convergence is one, for the test problem 1.

From the Tables 1 and 2, it is observed that using problem (3) with $\varepsilon = 0.125$, we are able to get the rate of order of convergence as one.

Table 2.Numerical rate of order of convergence, $\varepsilon = 0.125$

h	N	maximum absolute error	numerical rate P_N
2^{-3}	8	0.5	-
2^{-4}	16	1.328125E-01	1.912537159
2^{-5}	32	5.440533E-02	1.912537159
2^{-6}	64	2.493043E-02	1.287571034
2^{-7}	128	1.196232E-02	1.12584023
2^{-8}	256	5.862470E-03	1.06941055
2^{-9}	512	2.902383E-03	1.028916676
2^{-10}	1024	1.998993	1.014270724
2^{-11}	2048	1.972998	average rate $P = \frac{1}{7}\sum P_N = 1.334440501$

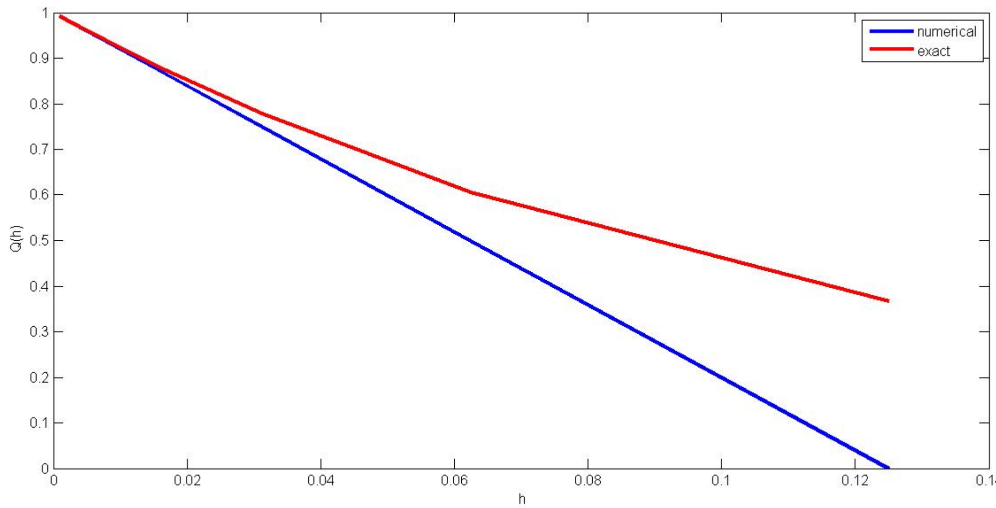
Order graph: The order graphs for the exact and numerical solutions of the test problem 1 is given in figure 13. The curve with blue colour refers to numerical solution and the curve with red colour refers to exact solution. It is a plot of h and Q(h). With respect to exact solution, $Q(h) = exp(-\frac{h}{\varepsilon})$ and with respect to the Euler's method $Q(h) = 1 - \frac{h}{\varepsilon}$. From figure 13, it is observed that, the curve with respect to numerical solution deviate downwards from the curve with respect to exact solution as time step progress. And, the order graph due to exact and numerical solutions are straight lines for $\varepsilon = 1$. But, for $\varepsilon = 0.125$ the order graph due to numerical solution is a straight line and for exact solution it is not a straight line.



(a). $\varepsilon = 1$

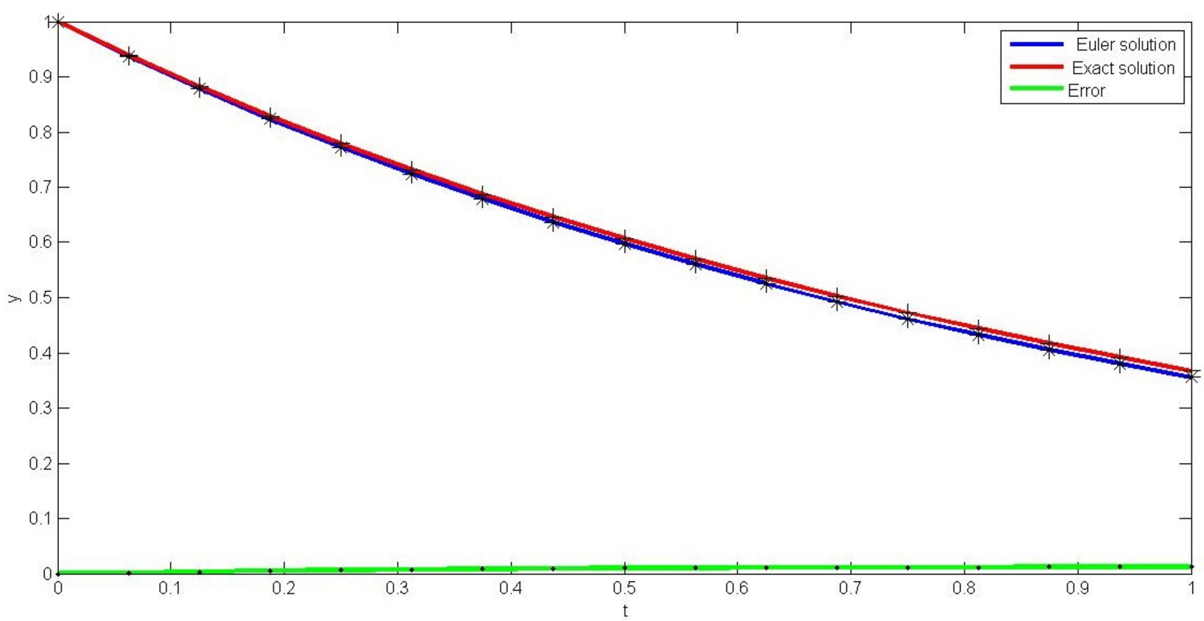
Numerical solution of singular perturbation problem: In figure 14, for the test problem 1, curves represent exact solution, numerical solution and the absolute error. It is observed from figure 14, that for $\varepsilon = 1$, as the numerical value deviate downwards from exact solution, the absolute error get deviate upwards from t-axis. If the absolute error is very closer to the t-axis, as time get increased, then the numerical solution come closer to the exact solution.

And, for $\varepsilon = 0.125$, the error curve is closer to te t-axis in the interval $(\varepsilon, 1)$ and in the interval $(0, \varepsilon)$ as the numerical solution deviate downwards from exact solution, the error curve deviate upwards. If the absolute error



(b). $\varepsilon = 0.125$

Figure 13.Stability regions of exact and numerical solutions, $\varepsilon = 0.5$.

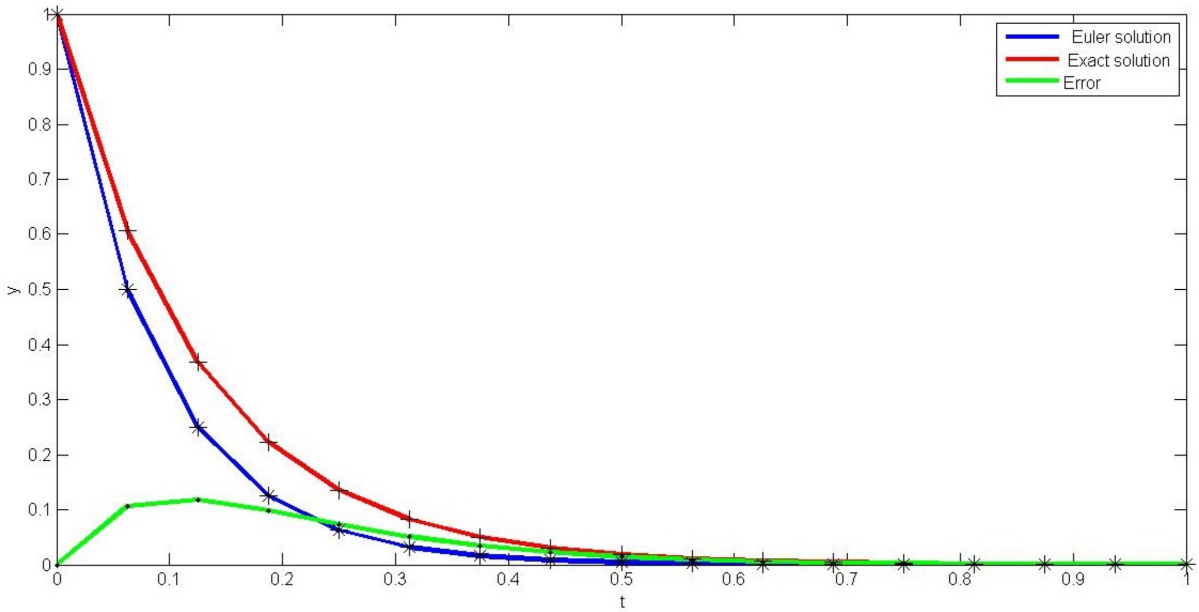


(a). $\varepsilon = 1$

is very closer to the t-axis in both the intervals $(0,\varepsilon)$ and $(\varepsilon,1)$ then, the numerical solution will be very closer to exact solution. In figure 15, for the test problem 2, curves represent exact solution, numerical solution and the absolute error. The observation is same as in test problem 1.

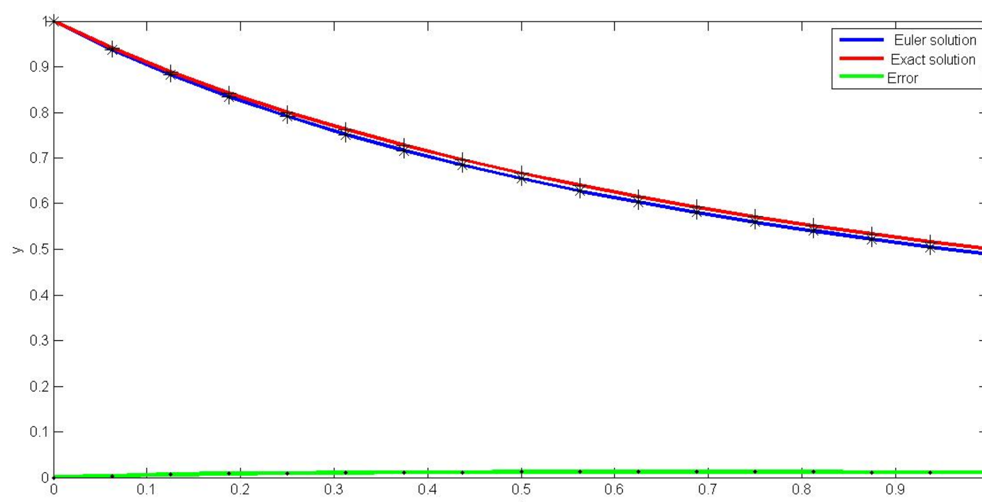
Numerical solution of singular perturbation problem with refined mesh: From the graphical results of figures 14 and 15, it is clear that, if the absolute error is very closer to the t-axis in both the intervals $(0,\varepsilon)$ and $(\varepsilon,1)$ then, the numericalsolution will be very closer to exact solution. From the stability region analysis of section 4, it is observed that, the stability region is a region inside contour with center $(\varepsilon,0)$ and radius ε. This motivates to take the interval [0,1] as the union of the intervals $[0,2\varepsilon]$ and $[2\varepsilon,1]$ and apply the numerical method as follows:

Using a step size h, partition the interval [0, 1] into N intervals. Note the value of N. Partition the interval $[0,2\varepsilon]$ into $\frac{N}{2}$ intervals and apply the numerical method



(b). $\varepsilon = 0.125$

Figure 14.Stability regions of exact and numerical solutions, $\varepsilon = 0.5$.



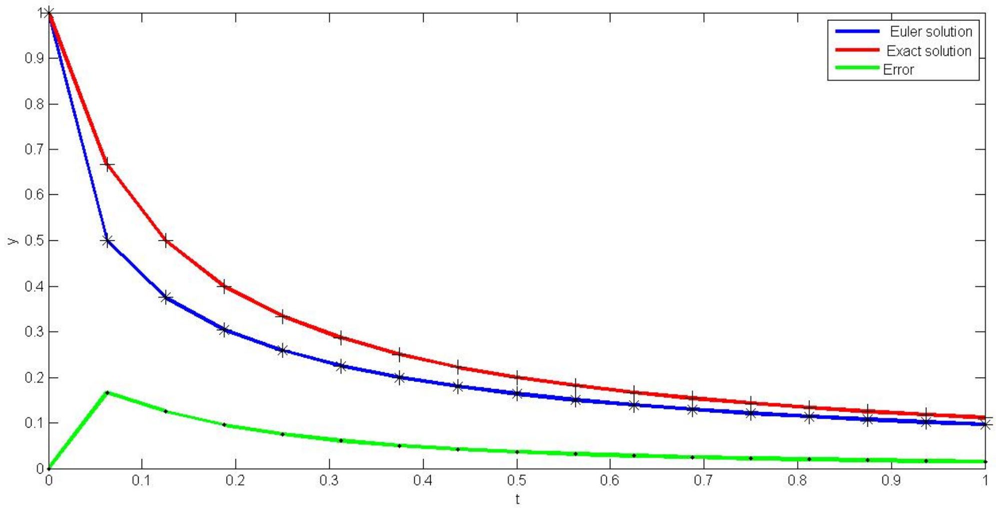
(a). $\varepsilon = 1$

$$y_{i+1} = y_i + \frac{h}{\varepsilon} f(t_i, y_i),\ i = 0, 1, 2, \cdots, \frac{N}{2} - 1,\ y_0 = \phi \tag{41}$$

Partition the interval $[2\varepsilon, 1]$ into $\frac{N}{2}$ intervals and apply the numerical method

$$y^*_{i+1} = y^*_i + \frac{h}{\varepsilon} f(t_i, y^*_i),\ i = 0, 1, 2, \cdots, \frac{N}{2} - 1,\ y^*_0 = y_{\frac{N}{2}} \tag{42}$$

Now, for h $\leq \varepsilon$, one can apply the methods (42) and (43) instead of (2) to solve the problem (1) for large values of h so that one can have more number of points in the interval $[0, 2\varepsilon]$.



(b). $\varepsilon = 0.125$

Figure 15.Exact and numerical solutions and absolute error, $h = \frac{1}{16}$

9. CONCLUSION

Calculating the shape of the unknown curve which starts at a given point and satisfies the given singularly perturbed differential equation is our problem. The shape of the unknown curve is obtained by Euler's method with order one convergence. Theoretical and numerical rates of order of convergence are derived. Stability regions, order star, order star finger region, relative stability region, relative absolute region and order graph are presented. The advantage of the numerical method is, it is explicit in nature and there is no need to iterate for numerical solution.

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