

RESEARCH ARTICLE

An Algorithm to find Definite Integrals using Simpson Rule

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ABSTRACT

An algorithmic approach to find definite single and double integrals using Simpson’s $\frac{1}{3}$ – rule is presented in this paper. This work will ensure the practical need of easy way of calculations with less computation time (run time) and storage space to engineers and scientists. It is observed that there is an opening to design a Simpson method which is different from the traditional Simpson method available in the literature for the numerical solution of ordinary differential equations. Numerical experiments were performed to show the validity of the algorithm. **Keywords:** Simpson rule, Single and double integrals, Newton’s interpolation, Less computation time, Storage space.

1. INTRODUCTION

In real time situations, scientists and engineers come across various practical difficulties in using both differential and integral equations as mathematical model for time dependent problems [1-4]. Many solutions to these problems do exist but with complicated steps it is further difficult to extend their ideas to higher dimensional problems. To make it easy, in this paper, a new approach of finding single and double definite integrals using Simpson rule are presented. In section 2, a definite single integral is derived and in section 3, a definite double integral is derived using Simpson rule. In section 4, a pictorial form of algorithm is given in two different forms for double integrals using Simpson rule and in section 5, numerical experiments are provided to show the performance of the single and double integrals using pictorial form of algorithm for Simpson’s $\frac{1}{3}$ – rule. The analysis is carried out by means of equations (1) to (19).

2. SINGLE DEFINITE INTEGRAL

In this section, a method for finding definite integral of $y=f(x)$ in $[a,b]$ is presented for a continuous function $f(x)$ using Simpson’s $\frac{1}{3}$ – rule. That is, to find the value of

$$\int_{x=a}^b f(x)dx \tag{1}$$

subdivide the interval $[a, b]$ using step size $h = x_{n+1} - x_n, n=0(1)N-1$ into N -subintervals and rewrite $[a, b]$ as union of $\frac{N}{2}$ - intervals of each length $2h$

$$[a, b] = \cup_{n=0}^{N-2} [x_n, x_{n+2}], n = 0(2), N - 2 \tag{2}$$

And the corresponding distribution table is shown in table 1.

Table 1.Distribution table for y

x	x ₀	x ₁	x ₂	.	.	.	x _{N-1}	x _N
y	y ₀	y ₁	y ₂	.	.	.	y _{N-1}	y _N

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Now from (1) and (2),

$$\int_{x=a}^b f(x)dx = \sum_{n=0}^{N-2} \int_{x_n}^{x_{n+2}} f(x)dx \tag{3}$$

Any $x \in [x_n, x_{n+2}]$, $n = 0(2)N - 2$ can be written in the form $x = x_n + uh$

As $x : x_n \rightarrow x_{n+2}$ the variable $u : 0 \rightarrow 2$. And so, equation (3) becomes

$$\int_{x=a}^b f(x)dx = h \sum_{n=0}^{N-2} \int_{u=0}^2 f(x_n + uh)du \tag{4}$$

Using Newton's interpolation formulae and $y = f(x)$ in (4),

$$y(x_n + uh) = y_n + uC_1\Delta y_n + uC_2\Delta^2 y_n + \dots \tag{5}$$

$$\int_{x=a}^b f(x)dx = h \sum_{n=0}^{N-2} \int_{u=0}^2 y(x_n + uh)du \tag{6}$$

Take only two terms of (5), equation (6) reduces to

$$\begin{aligned} \int_{x=a}^b f(x)dx &= h \sum_{n=0}^{N-2} \int_{u=0}^2 [y_n + uC_1\Delta y_n + uC_2\Delta^2 y_n]du \\ &= h \sum_{n=0}^{N-2} \int_{u=0}^2 \left[y_n + u\Delta y_n + \frac{u(u-1)}{2} \Delta^2 y_n \right] du \\ &= h \sum_{n=0}^{N-2} \left[uy_n + \frac{u^2}{2} \Delta y_n + \frac{1}{2} \left(\frac{u^3}{3} - \frac{u^2}{2} \right) \Delta^2 y_n \right]_{u=0}^{u=2} \\ &= h \sum_{n=0}^{N-2} \left[2y_n + \frac{2^2}{2} \Delta y_n + \frac{1}{2} \left(\frac{2^3}{3} - \frac{2^2}{2} \right) \Delta^2 y_n \right] \\ &= h \sum_{n=0}^{N-2} \left[2y_n + 2\Delta y_n + \frac{1}{3} \Delta^2 y_n \right] \\ &= \frac{h}{3} \sum_{n=0}^{N-2} [y_n + 4y_{n+1} + y_{n+2}], n = 0(2)N - 2 \end{aligned} \tag{7}$$

since $\Delta y_n = [y_{n+1} - y_n]$ and

$$\Delta^2 y_n = \Delta(y_{n+1} - y_n) = [y_{n+2} - y_{n+1}] - [y_{n+1} - y_n] = [y_n - 2y_{n+1} + y_{n+2}]$$

From (7), the definite integral is

$$\begin{aligned} \int_{x=a}^b f(x)dx &= \frac{h}{3} \sum_{n=0}^{N-2} [y_n + 4y_{n+1} + y_{n+2}], n = 0(2)N - 2 \\ &= \frac{h}{3} [y_0 + 4(y_1 + y_3 + \dots + y_{N-1}) + 2(y_2 + y_4 + \dots + y_{N-2}) + y_N] \\ &= \frac{h}{3} [y_0 + 4\sum_{n=1}^{N-1} y_n + 2\sum_{n=2}^{N-2} y_n + y_N] \end{aligned} \tag{8}$$

Remark $\int_{x=a}^b f(x)dx = \frac{h}{3}$ [(value of $y = f(x)$ at left end point $x = a$) + 4(leaving left end and right end points $x = a$ and $x = b$ find the sum of the values of $y = f(x)$ at odd suffixes) + 2(leaving left end and right end points $x = a$ and $x = b$ find the sum of the values of $y = f(x)$ at all even suffixes) + (value of $y = f(x)$ at right end point $x = b$)].

3. DOUBLE DEFINITE INTEGRAL

In this section, a method for finding definite integral of $z = f(x, y)$ in the domain $[a, b] \times [c, d]$ is presented for a continuous function $f(x, y)$ using Simpson's $\frac{1}{3}$ - rule. That is, to find the value of

$$\int_{x=a}^b \int_{y=c}^d f(x, y) dx dy \tag{9}$$

subdivide the interval $[a, b]$ using step size $h = x_{n+1} - x_n, n = 0(1)N - 1$ into $N - 1$ subintervals and rewrite $[a, b]$ as union of intervals of length $2h$

$$[a, b] = \cup_{n=0}^{N-2} [x_n, x_{n+2}], n = 0(2)N - 2 \tag{10}$$

and, subdivide the interval $[c, d]$ using step size $k = y_{m+1} - y_m, m = 0(1)M - 1$ into $M - 1$ subintervals and rewrite $[c, d]$ as union of intervals of length $2k$

$$[c, d] = \cup_{m=0}^{M-2} [y_m, y_{m+2}], m = 0(2)M - 2 \tag{11}$$

From (10) and (11), it follows that

$$[a, b] \times [c, d] = \cup_{n=0}^{N-2} \cup_{m=0}^{M-2} [x_n, x_{n+2}] \times [y_m, y_{m+2}] \tag{12}$$

for $n = 0(2)N - 2$ and $m = 0(2)M - 2$ and the corresponding distribution table is shown in table 2.

Table 2. Distribution table for $f(x, y)$

$x \backslash y$	y_0	y_1	y_2	.	y_j	.	y_{M-1}	y_M
x_0	$f(x_0, y_0)$	$f(x_0, y_1)$	$f(x_0, y_2)$		$f(x_0, y_j)$		$f(x_0, y_{M-1})$	$f(x_0, y_M)$
x_1	$f(x_1, y_0)$	$f(x_1, y_1)$	$f(x_1, y_2)$		$f(x_1, y_j)$		$f(x_1, y_{M-1})$	$f(x_1, y_M)$
x_2	$f(x_2, y_0)$	$f(x_2, y_1)$	$f(x_2, y_2)$		$f(x_2, y_j)$		$f(x_2, y_{M-1})$	$f(x_2, y_M)$
.								
x_i	$f(x_i, y_0)$	$f(x_i, y_1)$	$f(x_i, y_2)$		$f(x_i, y_j)$		$f(x_i, y_{M-1})$	$f(x_i, y_M)$
.								
x_{N-2}	$f(x_{N-2}, y_0)$	$f(x_{N-2}, y_1)$	$f(x_{N-2}, y_2)$		$f(x_{N-2}, y_j)$		$f(x_{N-2}, y_{M-1})$	$f(x_{N-2}, y_M)$
x_{N-1}	$f(x_{N-1}, y_0)$	$f(x_{N-1}, y_1)$	$f(x_{N-1}, y_2)$		$f(x_{N-1}, y_j)$		$f(x_{N-1}, y_{M-1})$	$f(x_{N-1}, y_M)$
x_N	$f(x_N, y_0)$	$f(x_N, y_1)$	$f(x_N, y_2)$		$f(x_N, y_j)$		$f(x_N, y_{M-1})$	$f(x_N, y_M)$

Now from (9) and (12),

$$\int_{x=a}^b \int_{y=c}^d f(x, y) dx dy = \sum_{n=0}^{N-2} \sum_{m=0}^{M-2} \int_{x_n}^{x_{n+2}} \int_{y_m}^{y_{m+2}} f(x, y) dx dy \tag{13}$$

Any $(x, y) \in [x_n, x_{n+2}] \times [y_m, y_{m+2}], n = 0(2)N - 2, m = 0(2)M - 2$ can be written in the form $x = x_n + uh, y = y_m + vk$. As $x : x_n \rightarrow x_{n+2}$ the variable $u : 0 \rightarrow 2$ and, as $y : y_m \rightarrow y_{m+2}$, the variable $v : 0 \rightarrow 2$ and so, equation (13) becomes

$$\int_{x=a}^b \int_{y=c}^d f(x, y) dx dy = hk \sum_{n=0}^{N-2} \sum_{m=0}^{M-2} \int_{u=0}^2 \int_{v=0}^2 f(x_n + uh, y_m + vk) du dv \tag{14}$$

Using Newton's interpolation formulae to $f(x_n + uh, y_m + vk)$ and taking first two terms

$$\int_{x=a}^b \int_{y=c}^d f(x, y) dx dy = hk \sum_{n=0}^{N-2} \sum_{m=0}^{M-2} \int_{u=0}^2 \left[\int_{v=0}^2 g(y_m + vk) dv \right] du \tag{15}$$

where $f(x_n + uh, y_m + vk) = g(y_m + vk)$ by treating first variable fixed. And using first three terms of Newton's interpolation and using equation(8)

$$g(y_m + vk) = g_m + vC_1\Delta g_m + vC_2\Delta^2 g_m + \dots, \tag{16}$$

$$\int_{x=a}^b \int_{y=c}^d f(x,y) dx dy = hk \sum_{n=0}^{N-2} \sum_{m=0}^{M-2} \int_{u=0}^2 \left[\int_{v=0}^2 (g_m + vC_1\Delta g_m + uC_2\Delta^2 g_m) dv \right] du \tag{17}$$

$$= \frac{hk}{3} \sum_{n=0}^{N-2} \int_{u=0}^2 [g_0 + 4\sum_{n=1}^{M-1} g_m + 2\sum_{n=2}^{M-2} g_m + g_M] du$$

$$= \frac{hk}{3} \sum_{n=0}^{N-2} \left[\int_{u=0}^2 g_0 du + 4\sum_{m=1}^{M-2} \int_{u=0}^2 g_m du + 2\sum_{m=2}^{M-2} \int_{u=0}^2 g_m du + \int_{u=0}^1 g_M du \right] \tag{18}$$

since, $f(x_n + uh, y_m + vk) = g(y_m + vk)$, hiding first variable, $g_0 = f(x_n + uh, y_0) = p(x_n + uh)$, hiding second variable, $g_m = f(x_n + uh, y_m + vk) = q(x_n + uh)$, hiding second variable, $m = odd$, $g_m = f(x_n + uh, y_m + vk) = q(x_n + uh)$, hiding second variable, $m = even$ and $g_M = f(x_n + uh, y_M) = r(x_n + uh)$, hiding second variable. Again using first three terms of Newton's interpolation to the functions p, q and r, and using (8) one have

$$\int_{x=a}^b \int_{y=c}^d f(x,y) dx dy = \frac{hk}{9} \left[\left(f(x_0, y_0) + 4\sum_{n=odd}^{N-1} f(x_n, y_0) + 2\sum_{n=even}^{N-2} f(x_n, y_0) + f(x_N, y_0) \right) + \left(f(x_0, y_M) + 4\sum_{n=odd}^{N-1} f(x_n, y_M) + 2\sum_{n=even}^{N-2} f(x_n, y_M) + f(x_N, y_M) \right) + \left(4\sum_{m=odd}^{M-1} f(x_0, y_m) + 2\sum_{m=even}^{M-2} f(x_0, y_m) \right) + \left(4\sum_{m=odd}^{M-1} f(x_N, y_m) + 2\sum_{m=even}^{M-2} f(x_N, y_m) \right) + \left(16\sum_{n=odd}^{N-1} \sum_{m=odd}^{M-1} f(x_n, y_m) + 8\sum_{n=odd}^{N-1} \sum_{m=even}^{M-2} f(x_n, y_m) + 8\sum_{n=even}^{N-2} \sum_{m=odd}^{M-1} f(x_n, y_m) + 4\sum_{n=even}^{N-2} \sum_{m=even}^{M-2} f(x_n, y_m) \right) \right] \tag{19}$$

Remark Two easy ways to find the double integral (19) are

$$\int_{x=a}^b \int_{y=c}^d f(x,y) dx dy = \frac{hk}{9} \left[\frac{3}{h} [\text{apply Simpson rule in the first row}] + \right.$$

$$\left. \frac{3}{h} [\text{apply Simpson rule on the last row}] + \right.$$

$(4[\text{sum of all entrices with odd suffixes in first column leaving first and last entrices}] +$

$2[\text{sum of all entrices with even suffixes in first column leaving first and last entrices}]) +$

$(4[\text{sum of all entrices with odd suffixes in last column leaving first and last entrices}] +$

$2[\text{sum of all entrices with even suffixes in last column leaving first and last entrices}]) +$

leaving first and last columns and leaving first and last rows find

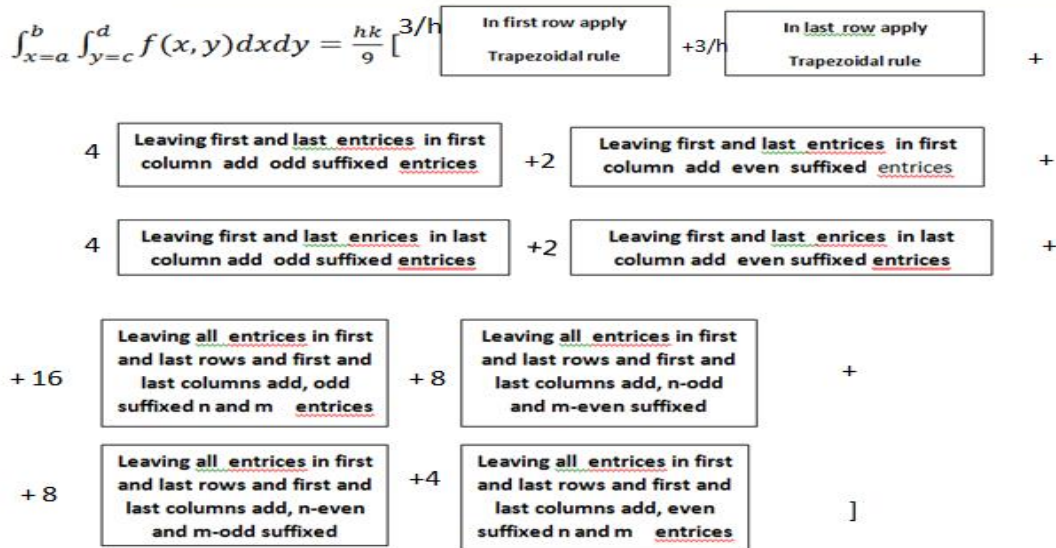
$$\left\{ (16[\text{sum of all entrics with odd suffixes of } n \text{ and } m] + 8[\text{sum of all entrics with odd suffixes of } n \text{ and even suffixes of } m] + 8[\text{sum of all entrics with even suffixes of } n \text{ and odd suffixes of } m] + 4[\text{sum of all entrics with even suffixes of } n \text{ and } m]) \right\}$$

and

$$\int_{x=a}^b \int_{y=c}^d f(x,y) dx dy = \frac{hk}{9} \left[\frac{3}{h} [\text{apply Simpson rule in the first column}] + \frac{3}{h} [\text{apply Simpson rule on the last column}] + (4[\text{sum of all entrics with odd suffixes in first row leaving first and last entrics}] + 2[\text{sum of all entrics with even suffixes in first row leaving first and last entrics}]) + (4[\text{sum of all entrics with odd suffixes in last row leaving first and last entrics}] + 2[\text{sum of all entrics with even suffixes in last row leaving first and last entrics}]) + \text{leaving first and last columns and leaving first and last rows find} \left\{ (16[\text{sum of all entrics with odd suffixes of } n \text{ and } m] + 8[\text{sum of all entrics with odd suffixes of } n \text{ and even suffixes of } m] + 8[\text{sum of all entrics with even suffixes of } n \text{ and odd suffixes of } m] + 4[\text{sum of all entrics with even suffixes of } n \text{ and } m]) \right\} \right]$$

4. AN ALGORITHM TO FIND THE DEFINITE DOUBLE INTEGRAL

This section presents an algorithm to evaluate the definite double integral using Simpson's $\frac{1}{3}$ - rule in two different pictorial forms instead of writing in steps. The pictorial forms of algorithm are



and

$$\int_{x=a}^b \int_{y=c}^d f(x, y) dx dy = \frac{hk}{9} \left[\begin{array}{l} \text{In first column apply Trapezoidal rule} \\ \text{In last column apply Trapezoidal rule} \\ +4 \left[\begin{array}{l} \text{Leaving first and last entries in first row add odd suffixed entries} \\ \text{Leaving first and last entries in first row add even suffixed entries} \end{array} \right] \\ +4 \left[\begin{array}{l} \text{Leaving first and last entries in last row add odd suffixed entries} \\ \text{Leaving first and last entries in last row add even suffixed entries} \end{array} \right] \\ +16 \left[\begin{array}{l} \text{Leaving all entries in first and last rows and first and last columns add, odd suffixed n and m entries} \\ \text{Leaving all entries in first and last rows and first and last columns add, n-odd and m-even suffixed} \end{array} \right] \\ +8 \left[\begin{array}{l} \text{Leaving all entries in first and last rows and first and last columns add, n-even and m-odd suffixed} \\ \text{Leaving all entries in first and last rows and first and last columns add, even suffixed n and m entries} \end{array} \right] \end{array} \right]$$

Using these pictorial forms, in next section, some single and double integrals are evaluated. Above two pictorial forms will generate the same value.

5. NUMERICAL EXPERIMENT

In this section, five single and four double integrals using Simpson's $\frac{1}{3}$ -rule are evaluated using remarks in section 2 and section 3 and pictorial form of algorithm in section 4. The step size used in all problems is $h = \frac{1}{4}$ and $k = \frac{1}{4}$.

Problem 1

Evaluate $\int_{x=0}^1 e^{-x^2} dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	1	0.9394	0.7788	0.5697	0.3678

Using remarks in section 2,

$$\int_{x=0}^1 e^{-x^2} dx = \frac{0.25}{3} [1 + 4(0.9394 + 0.5697) + 2(0.7788) + 0.3678] = 0.61638 \text{ units.}$$

Problem 2

Evaluate $\int_{x=0}^1 e^{-x} dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	1	0.778	0.606	0.472	0.368

Using remarks in section 2,

$$\int_{x=0}^1 e^{-x} dx = \frac{0.25}{3} [1 + 4(0.778 + 0.472) + 2(0.606) + 0.368] = 0.6316 \text{ units.}$$

Problem 3

Evaluate $\int_{x=0}^1 x^2 dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	0	0.0635	0.25	0.5625	1

Using remarks in section 2,

$$\int_{x=0}^1 x^2 dx = \frac{0.25}{3} [0 + 4(0.0655 + 0.5625) + 2(0.25) + 1] = 0.33433 \text{ units.}$$

Problem 4

Evaluate $\int_{x=0}^1 \sin(x) dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	0	0.2474	0.4794	0.6816	0.8414

Using remarks in section 2,

$$\int_{x=0}^1 \sin(x) dx = \frac{0.25}{3} [0 + 4(0.2474 + 0.6816) + 2(0.4794) + 0.8414] = 0.4596 \text{ units.}$$

Problem 5

Evaluate $\int_{x=0}^1 \cos(x) dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	1	0.9689	0.8775	0.7316	0.5403

Using remarks in section 2,

$$\int_{x=0}^1 \cos(x) dx = \frac{0.25}{3} [1 + 4(0.9689 + 0.7316) + 2(0.8775) + 0.5403] = 0.84144 \text{ units.}$$

Problem 6

Evaluate $\int_{x=0}^1 \int_{y=0}^1 e^{-x^2} e^{-y^2} dx dy$

The corresponding distribution table is

x/y	0	0.25	0.5	0.75	1.0
0	1	0.9394	0.7788	0.5697	0.3678
0.25	0.9394	0.8824	0.7316	0.5351	0.3455
0.5	0.7788	0.7316	0.6065	0.4436	0.2864
0.75	0.5697	0.5351	0.4436	0.3246	0.2095
1.0	0.3678	0.3455	0.2864	0.2095	0.1353

Using the pictorial form of algorithm

$$\int_{x=0}^1 \int_{y=0}^1 e^{-x^2} e^{-y^2} dx dy =$$

$$\frac{0.25 \times 0.25}{9} [7.3965 + 3.2989 + 6.8152 + 2.7928 + 36.4832 + 9.4016 + 9.4016 + 2.426] = 0.55477 \text{sq. units.}$$

Problem 7

Evaluate $\int_{x=0}^1 \int_{y=0}^1 e^{-x} e^{-y} dx dy$

The corresponding distribution table is

x/y	0	0.25	0.5	0.75	1.0
0	1	0.778	0.606	0.472	0.368
0.25	0.778	0.606	0.472	0.368	0.286
0.5	0.606	0.472	0.368	0.286	0.223
0.75	0.472	0.368	0.286	0.223	0.173
1.0	0.368	0.286	0.223	0.173	0.125

Using the pictorial form of algorithm

$$\int_{x=0}^1 \int_{y=0}^1 e^{-x} e^{-y} dx dy =$$

$$\frac{0.25 \times 0.25}{9} [7.58 + 2.775 + 6.212 + 2.282 + 25.072 + 6.064 + 6.064 + 1.472] = 0.3994 \text{ sq. units.}$$

Problem 8

Evaluate $\int_{x=0}^1 \int_{y=0}^1 x^2 y^2 dx dy$

The corresponding distribution table is

x/y	0	0.25	0.5	0.75	1.0
0	0	0	0	0	0
0.25	0	0.0039	0.0156	0.0351	0.0625
0.5	0	0.0156	0.0625	0.1406	0.25
0.75	0	0.0351	0.1406	0.3164	0.5625
1.0	0	0.0625	0.25	0.5625	1

Using the pictorial form of algorithm

$$\int_{x=0}^1 \int_{y=0}^1 x^2 y^2 dx dy = \frac{0.25 \times 0.25}{9} [0 + 3 + 0 + 3 + 6.248 + 1.2496 + 1.2496 + 0.25] = 0.1042 \text{ sq. units.}$$

Problem 9

Evaluate $\int_{x=0}^1 \int_{y=0}^1 \sin(x)\cos(y) dx dy$
 The corresponding distribution table is

x/y	0	0.25O	0.5E	0.75O	1.0
0	0	0	0	0	0
0.25O	0.2174	0.2397	0.2170	0.1809	0.1338O
0.5E	0.4794	0.4644	0.4206	0.3504	0.2590E
0.75O	0.6816	0.6604	0.5981	0.4986	0.3682O
1.0	0.8414	0.8132	0.7383	0.6155	0.4546

Using the pictorial form of algorithm

$$\int_{x=0}^1 \int_{y=0}^1 \sin(x)\cos(y) dx dy = \frac{0.25 \times 0.25}{9} [5.3962 + 2.9805 + 0 + 7.252 + 25.2736 + 6.5208 + 6.5184 + 1.6824] = 0.3862 \text{ sq. units.}$$

It is observed that the numerical solution using traditional Simpson’s $\frac{1}{3}$ – rule for ordinary differential equation do not match with the definite integral of a single integral of a function f(x) of section 2. This motivates to design a new Simpson’s $\frac{1}{3}$ – rule for the numerical solution of ordinary differential equations using the proof used in this paper.

6. CONCLUSION

In this paper, an algorithm for Simpson’s $\frac{1}{3}$ – rule is presented, theoretically proved, pictorially described and easy to implement. The approach applied to single integral is extended to double integral. This approach can be extended to higher dimensions. Simpson’s $\frac{1}{3}$ – rule derivation of this paper gives an opening to find another Simpson’s $\frac{1}{3}$ – rule which is different from the traditional Simpson’s $\frac{1}{3}$ – rule to solve stiff and non-stiff ordinary and partial differential equations. Research is going on in this direction by the author.

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