DJ Journal of Engineering and Applied Mathematics, Vol. 3(2) 2017, pp. 1-8

RESEARCH ARTICLE

An Algorithm to find Definite Integrals

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Received- 10 February 2017, Revised- 29 April 2017, Accepted- 10 May 2017, Published- 19 May 2017

ABSTRACT

An algorithm to find definite single and double integrals using Trapezoidal $\frac{1}{2}$-rule is presented in this paper. This work will fulfill the need of easy way of calculations with less computation time (run time) and storage space to engineers and scientists. It is observed that there is an opening to design a trapezoidal method which is entirely different from the traditional trapezoidal method for the numerical solution of ordinary differential equations. Numerical experiments were performed to show the validity of the algorithm.
Keywords: Trapezoidal rule, Single and double integrals, Newton's interpolation, Numerical solution, Differential equations.

1. INTRODUCTION

In real time situations, scientists and engineers come across both differential and integral equations as mathematical model for time dependent problems[1-4]. Many solutions to these problems exists but with complicated steps further no one can extend their ideas to higher dimensional problems. To make it easy, in this paper, a new approach of finding single and double definite integrals are presented. In section 2, a definite single integral is derived and in section 3, a definite double integral is derived. In section 4, a pictorial form of algorithm is given in two different forms for double integrals and in section 5, numerical experiments are provide to show the performance of the single and double integrals using pictorial form of algorithm. The analysis is carried out by means of equations (1) to (19).

2. SINGLE DEFINITE INTEGRAL

In this section, a method for finding definite integral of $y = f(x)$ in $[a,b]$ is presented for a continuous function $f(x)$. That is, to find the value of

$$\int_{x=a}^{b} f(x)\,dx \tag{1}$$

subdivide the interval $[a,b]$ using step size $h = x_{n+1} - x_n$, $n = 0(1)N-1$ into $N-$subintervals and so

$$[a,b] = \cup_{n=0}^{N-1}[x_n, x_{n+1}] \tag{2}$$

And the corresponding distribution table is given in table 1.

Table 1.Distribution table for y

x	x_0	x_1	x_2	.	.	.	x_{N-1}	x_N
y	y_0	y_1	y_2	.	.	.	y_{N-1}	y_N

Now from (1) and (2),

$$\int_{x=a}^{b} f(x)dx = \Sigma_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} f(x)dx \tag{3}$$

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Double blind peer review under responsibility of DJ Publications
https://dx.doi.org/10.18831/djmaths.org/2017021001

Any $x\varepsilon[x_n, x_{n+1}]$, $n = 0(1)N-1$ can be written in the form $x = x_n + uh$. As $x : x_n \to x_{n+1}$ the variable $u : 0 \to 1$. And so, equation (3) becomes

$$\int_{x=a}^{b} f(x)dx = h\Sigma_{n=0}^{N-1} \int_{u=0}^{1} f(x_n + uh)du \tag{4}$$

Using Newton's interpolation formulae and $y - f(x)$ in (4),

$$y(x_n + uh) = y_n + uC_1\Delta y_n + uC_2\Delta^2 y_n + ... \tag{5}$$

$$\int_{x=a}^{b} f(x)dx = h\Sigma_{n=0}^{N-1} \int_{u=0}^{1} y(x_n + uh)du \tag{6}$$

Taking only two terms of (5), equation (6) reduces to

$$\int_{x=a}^{b} f(x)dx = h\Sigma_{n=0}^{N-1} \int_{u=0}^{1} [y_n + uC_1\Delta y_n]du$$

$$= h\Sigma_{n=0}^{N-1} \int_{u=0}^{1} [y_n + u\Delta y_n]du$$

$$= \frac{h}{2}\Sigma_{n=0}^{(N-1)} [y_{n+1} + y_n] \tag{7}$$

since $\Delta y_n = [y_{n+1} - y_n]$ and $uC_1 = u$. From (7), the definite integral is

$$\int_{x=a}^{b} f(x)dx = \frac{h}{2}\Sigma_{n=0}^{N-1} [y_n + y_{n+1}]$$

$$= \frac{h}{2}[y_0 + 2(y_1 + y_2 + \ldots + y_{N-1}) + y_N] \tag{8}$$

Remark $\int_{x=a}^{b} f(x)dx = \frac{h}{2}$ [(value of f(x) at left end point x=a) + 2(leaving left end and right end points x=a and x=b, find the sum of the values of f(x) at all other points) + (value of f(x) at left end point x=b)].

3. DOUBLE DEFINITE INTEGRAL

In this section, a method for finding definite integral of $z = f(x, y)$ in the domain $[a, b]X[c, d]$ is presented for a continuous function $f(x, y)$. That is, to find the value of

$$\int_{x=a}^{b} \int_{y=c}^{d} f(x, y)dxdy \tag{9}$$

subdivide the interval [a, b] using step size $h = x_{n+1} - x_n, n = 0(1)N-1$ into $N-$subintervals and so

$$[a, b] = \cup_{n=0}^{N-1} [x_n, x_{n+1}]. \tag{10}$$

And, subdivide the interval [c, d] using step size $k = y_{m+1} - y_m$, $m = 0(1)M-1$ into M-subintervals and so

$$[c, d] = \cup_{m=0}^{M-1} [y_m, y_{m+1}]. \tag{11}$$

From (10) and (11), it follows that

$$[a, b]X[c, d] = \cup_{n=0}^{N-1} \cup_{m=0}^{m-1} [x_n, x_{n+1}]X[y_m, y_{m+1}] \tag{12}$$

And the corresponding distribution table is shown in table 2.

Table 2.Distribution table for f(x,y)

$x\backslash y$	y_0	y_1	y_2	.	y_j	.	y_{M-1}	y_M
x_0	$f(x_0,y_0)$	$f(x_0,y_1)$	$f(x_0,y_2)$		$f(x_0,y_j)$		$f(x_0,y_{M-1})$	$f(x_0,y_M)$
x_1	$f(x_1,y_0)$	$f(x_1,y_1)$	$f(x_1,y_2)$		$f(x_1,y_j)$		$f(x_1,y_{M-1})$	$f(x_1,y_M)$
x_2	$f(x_2,y_0)$	$f(x_2,y_1)$	$f(x_2,y_2)$		$f(x_2,y_j)$		$f(x_2,y_{M-1})$	$f(x_2,y_M)$
.								
x_i	$f(x_i,y_0)$	$f(x_i,y_1)$	$f(x_i,y_2)$		$f(x_i,y_j)$		$f(x_i,y_{M-1})$	$f(x_i,y_M)$
.								
x_{N-2}	$f(x_{N-2},y_0)$	$f(x_{N-2},y_1)$	$f(x_{N-2},y_2)$		$f(x_{N-2},y_j)$		$f(x_{N-2},y_{M-1})$	$f(x_{N-2},y_M)$
x_{N-1}	$f(x_{N-1},y_0)$	$f(x_{N-1},y_1)$	$f(x_{N-1},y_2)$		$f(x_{N-1},y_j)$		$f(x_{N-1},y_{M-1})$	$f(x_{N-1},y_M)$
x_N	$f(x_N,y_0)$	$f(x_N,y_1)$	$f(x_N,y_2)$		$f(x_N,y_j)$		$f(x_N,y_{M-1})$	$f(x_N,y_M)$

Now from (9) and (12),

$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)dxdy = \Sigma_{n=0}^{N-1}\Sigma_{m=0}^{M-1}\int_{x_n}^{x_{n+1}}\int_{y_m}^{y_{m+1}} f(x,y)dxdy \tag{13}$$

Any $(x,y)\varepsilon[x_n,x_{n+1}]X[y_m,y_{m+1}]$, $n=0(1)N-1$, $m=0(1)M-1$ can be written in the form $x=x_n+uh$ and $y=y_m+vk$. As $x: x_n\to x_{n+1}$, the variable $u: 0\to 1$ and, as $y: y_m\to y_{m+1}$ the variable $v: 0\to 1$.

So, equation (13) becomes

$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)dxdy = hk\Sigma_{n=0}^{N-1}\Sigma_{m=0}^{M-1}\int_{u=0}^{1}\int_{v=0}^{1} f(x_n+uh,y_m+vk)dudv \tag{14}$$

Using Newton's interpolation formulae to $f(x_n+uh,y_m+vk.)$ and taking first two terms

$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)dxdy = hk\Sigma_{n=0}^{N-1}\Sigma_{m=0}^{M-1}\int_{u=0}^{1}\left[\int_{v=0}^{1} g(y_m+vk)dv\right]du \tag{15}$$

where $f(x_n+uh,y_m+vk)=g(y_m+vk)$ by treating first variable fixed. And using first two terms of Newton's interpolation

$$g(y_m+vk) = g_m + vC_1\Delta g_m + vC_2\Delta^2 g_n + \ldots, \tag{16}$$

$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)dxdy$$

$$= hk\Sigma_{n=0}^{N-1}\Sigma_{m=0}^{M-1}\int_{u=0}^{1}\left[\int_{v=0}^{1} g_m + vC_1\Delta g_m dv\right]du \tag{17}$$

$$= \frac{hk}{2}\Sigma_{n=0}^{N-1}\int_{u=0}^{1}\left[\Sigma_{m=0}^{M-1}[g_m+g_{m+1}\right]du$$

$$= \frac{hk}{2}\Sigma_{n=0}^{N-1}\int_{u=0}^{1}[g_0+2\Sigma_{m=1}^{M-1}g_m+g_M]du$$

$$= \frac{hk}{2}\Sigma_{n=0}^{N-1}\left[\int_{u=0}^{1}g_0du + 2\Sigma_{m=1}^{M-1}\int_{u=0}^{1}g_mdu + \int_{u=0}^{1}g_Mdu\right] \tag{18}$$

Since, $f(x_n+uh,y_m+vk)=g(y_m+vk)$,

$g_0 = f(x_n + uh, y_0) = p(x_n + uh)$, fixing second variable
$g_m = f(x_n + uh, y_m + vk) = q(x_n + uh)$, fixing second variable and
$g_M = f(x_n + uh, y_M) = r(x_n + uh)$, fixing second variable
Again using first two terms of Newton's interpolation to the functions p, q and r , one have

$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)dxdy = \frac{hk}{4}[(f(x_0,y_0)+2\sum_{n=1}^{N-1} f(x_n,y_0)+f(x_N,y_0))+(f(x_0,y_M)+2\Sigma_{n=1}^{N-1} f(x_n,y_M) + f(x_N,y_M))+2\Sigma_{m=1}^{M-1} f(x_0,y_m)+2\Sigma_{m=1}^{M-1} f(x_N,y_m)+4\Sigma_{n=1}^{N-1}{}_{m=1}^{M-1} f(x_n,y_m)] \tag{19}$$

Remark Two easy ways to find the double integral (19) are

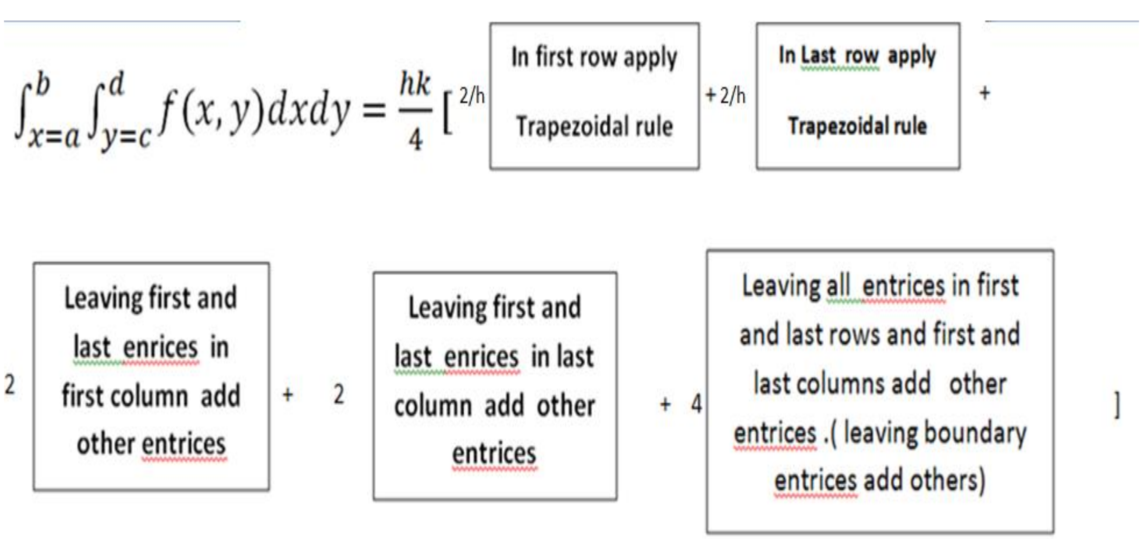
$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)dxdy = \frac{hk}{4}[\frac{2}{h}\lfloor apply\ Trapezoidal\ in\ the\ first\ row\rfloor + \frac{2}{h}\lfloor apply\ Trapezoidal\ on\ the\ last\ row\rfloor + 2\lfloor\ sum\ of\ all\ entrices\ in\ the\ first\ column\ leaving\ first\ and\ last\ entrices\rfloor + 2\lfloor\ sum\ of\ all\ entrices\ in\ the\ last\ column\ leaving\ first\ and\ last\ entrices\rfloor + 4\lfloor\ sum\ of\ all\ entrices\ leaving\ first\ and\ last\ columns\ and\ leaving\ first\ and\ lastrows\rfloor]$$

and

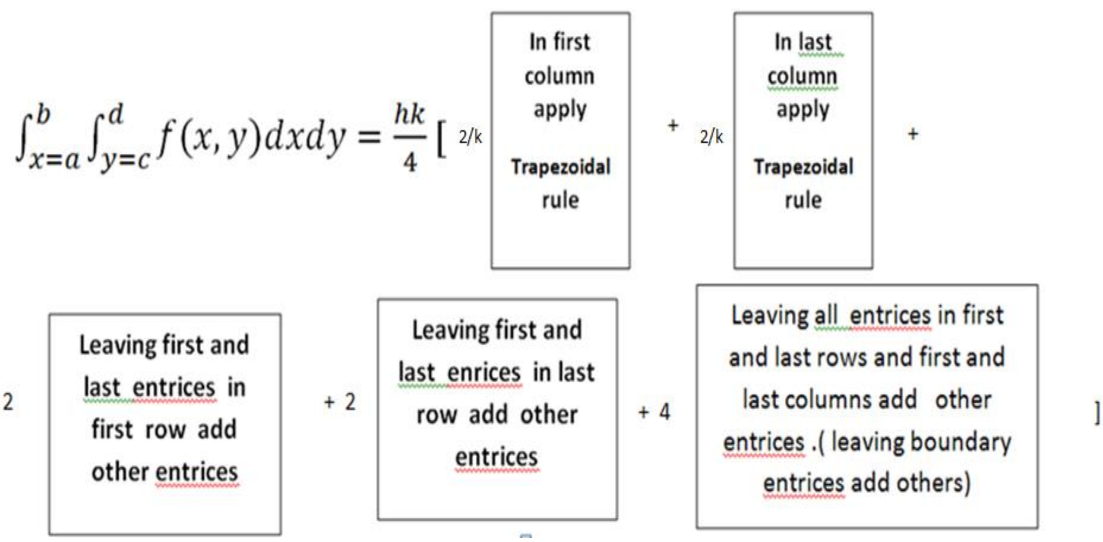
$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)dxdy = \frac{hk}{4}[\frac{2}{k}\lfloor apply\ Trapezoidal\ in\ the\ first\ column\rfloor + \frac{2}{h}\lfloor apply\ Trapezoidal\ on\ the\ last\ column\rfloor + 2\lfloor\ sum\ of\ all\ entrices\ in\ the\ first\ row\ leaving\ first\ and\ last\ entrices\rfloor + 2\lfloor\ sum\ of\ all\ entrices\ in\ the\ last\ row\ leaving\ first\ and\ last\ entrices\rfloor + 4\lfloor\ sum\ of\ all\ entrices\ leaving\ first\ and\ last\ columns\ and\ leaving\ first\ and\ lastrows\rfloor]$$

4. AN ALGORITHM TO EVALUATE THE DEFINITE DOUBLE INTEGRAL

This section presents an algorithm to evaluate the definite double integral in two different pictorial forms instead of writing in steps. The pictorial forms of the algorithm are



and



Using these pictorial forms, in next section, some single and double integrals are evaluated. Above two pictorial forms will generate the same value.

5. NUMERICAL EXPERIMENT

In this section, five single and four double integrals are evaluated using remarks in section 2 and section 3 and pictorial form of algorithm in section 4. The step size used in all problems is $h = \frac{1}{4}$ and $k = \frac{1}{4}$.

Problem 1

Evaluate $\int_{x=0}^{1} e^{-x^2} dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	1	0.9394	0.7788	0.5697	0.3678

Using remarks in section 2,

$$\int_{x=0}^{1} e^{-x^2} dx = \frac{0.25}{2}[1 + 2(0.9394 + 0.7788 + 0.5697) + 0.3678] = 0.74295 \text{ units.}$$

Problem 2

Evaluate $\int_{x=0}^{1} e^{-x} dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	1	0.778	0.606	0.472	0368

Using remarks in section 2,

$$\int_{x=0}^{1} e^{-x} dx = \frac{0.25}{2}[1+2(0.778+0.606+0.472)+0.368] = 0.635 \text{ units.}$$

Problem 3

Evaluate $\int_{x=0}^{1} x^2 dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	0	0.0635	0.25	0.5625	1

Using remarks in section 2,

$$\int_{x=0}^{1} x^2 dx = \frac{0.25}{2}[0+2(0.0655+0.25+0.5625)+1] = 0.34375 \text{ units.}$$

Problem 4

Evaluate $\int_{x=0}^{1} sin(x) dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	0	0.2474	0.4794	0.6816	0.8414

Using remarks in section 2,

$$\int_{x=0}^{1} sin(x) dx = \frac{0.25}{2}[0+2(1.4084)+0.8414] = 0.4572 \text{ units.}$$

Problem 5

Evaluate $\int_{x=0}^{1} cos(x) dx$

The corresponding distribution table is

x	0	0.25	0.5	0.75	1.0
y	1	0.9689	0.8775	0.7316	0.5403

Using remarks in section 2,

$$\int_{x=0}^{1} cos(x) dx = \frac{0.25}{2}[1+2(2.598)+0.5403] = 0.84203 \text{ units.}$$

Problem 6

Evaluate $\int_{x=0}^{1} \int_{y=0}^{1} e^{-x^2} e^{-y^2} dxdy$

The corresponding distribution table is

x\y	0	0.25	0.5	0.75	1.0
0	1	0.9394	0.7788	0.5697	0.3678
0.25	0.9394	0.8824	0.7316	0.5351	0.3455
0.5	0.7788	0.7316	0.6065	0.4436	0.2864
0.75	0.5697	0.5351	0.4436	0.3246	0.2095
1.0	0.3678	0.3455	0.2864	0.2095	0.1353

Using the pictorial form of algorithm,

$\int_{x=0}^{1}\int_{y=0}^{1} e^{-x^2}e^{-y^2}dxdy = \frac{0.25x0.25}{4}[5.9436+2.1861+2(2.2879)+2(0.8414)+4(5.2341)] = 0.5519$ sq. units.

Problem 7

Evaluate $\int_{x=0}^{1}\int_{y=0}^{1} e^{-x}e^{-y}dxdy$

The corresponding distribution table is

x\y	0	0.25	0.5	0.75	1.0
0	1	0.778	0.606	0.472	0.368
0.25	0.778	0.606	0.472	0.368	0.286
0.5	0.606	0.472	0.368	0.286	0.223
0.75	0.472	0.368	0.286	0.223	0.173
1.0	0.368	0.286	0.223	0.173	0.125

Using the pictorial form of algorithm,

$\int_{x=0}^{1}\int_{y=0}^{1} e^{-x}e^{-y}dxdy = \frac{0.25x0.25}{4}[4.46+1.86+2(1.856)+2(0.682)+4(3.449)] = 0.0213$ sq. units.

Problem 8

Evaluate $\int_{x=0}^{1}\int_{y=0}^{1} x^2y^2dxdy$

The corresponding distribution table is

x\y	0	0.25	0.5	0.75	1.0
0	0	0	0	0	0
0.25	0	0.0039	0.0156	0.0351	0.0625
0.5	0	0.0156	0.0625	0.1406	0.25
0.75	0	0.0351	0.1406	0.3164	0.5625
1.0	0	0.0625	0.25	0.5625	1

Using the pictorial form of algorithm,

$$\int_{x=0}^{1}\int_{y=0}^{1} x^2y^2dxdy = \frac{0.25x0.25}{4}[0+2.75+2(0)+2(1.75)4(0.7654)] = 0.1454 \text{ sq. units.}$$

Problem 9

Evaluate $\int_{x=0}^{1}\int_{y=0}^{1} sin(x)cos(y)dxdy$

The corresponding distribution table is

x\y	0	0.25	0.5	0.75	1.0
0	0	0	0	0	0
0.25	0.2174	0.2397	0.2170	0.1809	0.1338
0.5	0.4794	0.4644	0.4206	0.3504	0.2590
0.75	0.6816	0.6604	0.5981	0.4986	0.3682
1.0	0.8414	0.8132	0.7383	0.6155	0.4546

Using the pictorial form of algorithm,

$$\int_{x=0}^{1}\int_{y=0}^{1} sin(x)cos(y)dxdy = \frac{0.25x0.25}{4}[3.5082+1.9766+2(0)+2(2.1858)+4(3.6501)] = 0.3820 \text{ sq. units.}$$

It is observed that the numerical solution using traditional trapezoidal method for ordinary differential equation do not match with the definite integral of a single integral of a function $f(x)$ of section 2. This motivates to design a new trapezoidal method for the numerical solution of ordinary differential equations using the method of proof in this paper.

6. CONCLUSION

An algorithm is presented in this paper, which is theoretically proved, pictorially described and is very easy to implement. The approach applied to single integral is extended to double integral too. This approach can be extended to higher dimensions also. Trapezoidal method derivation of this paper gives an opening to find another trapezoidal method which is different from the traditional trapezoidal method to solve stiff and non-stiff ordinary and partial differential equations. Research is going on in this direction by the author of this paper.

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