DJ Journal of Engineering and Applied Mathematics, Vol. 3(1) 2017, pp. 13-20

RESEARCH ARTICLE

Properties of pre-$\mathcal{I}$-open sets *

*R Sajuntha[1]

[1]Department of Mathematics, Ponjesly College of Engineering, Nagercoil- 629 003, Tamil Nadu, India.

Received- 10 November 2016, Revised- 25 December 2016, Accepted- 20 January 2017, Published- 31 January 2017

ABSTRACT

In 1982, [5] have defined the notion of preopen sets. Ouite recently in 2002, [6] has defined pre neighbourhoods, pre-interior point, prelimit point, prederived set and prefrontier of set. For these sets we define the notion of pre-$\mathcal{I}$-interior point, pre-$\mathcal{I}$-limit point, and pre-$\mathcal{I}$-frontier of a set using pre-$\mathcal{I}$-open sets and pre-$\mathcal{I}$-closed sets in ideal topological space. We also obtained a necessary and sufficient condition for an ideal topological space to be a pre-$\mathcal{I}$-T_1 space.

Keywords: Ideal spaces, Pre-$\mathcal{I}$-open sets, Pre-$\mathcal{I}$-closed sets, Pre-$\mathcal{I}$-T_1 space, Pre-$\mathcal{I}$-interior point, Pre-$\mathcal{I}$-frontier of a set.

1. INTRODUCTION AND PRELIMINARIES

The subject of ideals in topological space has been studied by [4] and [7-9]. An ideal $\mathcal{I}$ on a topological space (X,τ) is a non empty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [4] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X, A^*(\tau,\mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}\}$, for every $\{U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $cl^*()$ for a topology $\tau^*(X,\tau)$, called the $\star-$topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{I},\tau)$ [7, 8]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal space. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}-$open [3] if $A \subset int(A^*)$. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be pre$-\mathcal{I}-$open [1] if $A \subset int(cl^*(A))$. The complement of pre$-\mathcal{I}-$open set is called pre$-\mathcal{I}-$closed. The family of all pre$-\mathcal{I}-$open sets in $(X,\tau,\mathcal{I})$ is denoted by $P\mathcal{I}O(X,\tau,\mathcal{I})$ or simply $P\mathcal{I}O(X)$. Clearly $\tau \subset P\mathcal{I}O(X)$. The largest pre$-\mathcal{I}-$open set contained in A, denoted by $p\mathcal{I}int(A)$, is called the pre$-\mathcal{I}-$interior of A. The smallest pre$-\mathcal{I}-$closed set containing A, denoted by $p\mathcal{I}cl(A)$, is called the pre$-\mathcal{I}-$closure of A. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\star-$dense [2] if $cl^*(A) = X$.

2. SOME MORE RESULTS ON PRE$-\mathcal{I}-$OPEN SETS

Theorem 2.1

If $A \subset X$ is $\star-$dense in $(X,\tau,\mathcal{I})$, then A is a pre$-\mathcal{I}-$open set.

Proof

Suppose A is $\star-$dense in X. Then $cl^*(A)) = X$ which implies that $int(cl^*(A)) = int(X) = X$. Therefore, $A \subset int(cl^*(A))$. Hence A is pre$-\mathcal{I}-$open.

The converse of the Theorem 2.1 need not be true as shown by the following Example 2.2.

Example 2.2

Consider the ideal space $(X,\tau,\mathcal{I})$ where $X = \{a,b,c\}, \tau = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$ and $\mathcal{I} = \{\emptyset,\{a\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. Let $A = \{b\}$. Then $cl^*(A) = \{b,c\}$ and $int(cl^*\{b\}) = b$. Therefore, A is pre$-\mathcal{I}-$open. But $cl^*(\{b\}) = \{b,c\} \neq X$. Hence A is not $\star-$dense.

*Corresponding author. Tel.: +919445408667
Email address: r.sajuntha@gmail.com (R.Sajuntha)
Double blind peer review under responsibility of DJ Publications
http://dx.doi.org/10.18831/djmaths.org/2017011002

Theorem 2.3

Let A be a subset of an open set B in an ideal topological space $(X,\tau,\mathcal{I})$ such that $A \subset B \subset cl^*(A)$. Then A is a pre$-\mathcal{I}-$open set.

Proof

Given that B is an open set in X so that, $B = int(B)$. Now $A \subset B \subset cl^*(A)$ implies that $int(A) \subset int(B) \subset int(cl^*(A))$ which inturn implies $int(A) \subset B \subset int(cl^*(A))$. Therefore, $A \subset int(cl^*(A))$. Hence A is pre$-\mathcal{I}-$open in X.

Theorem 2.4

In an ideal topological space $(X,\tau,\mathcal{I})$, arbitrary union of pre$-\mathcal{I}-$open sets is a pre$-\mathcal{I}-$open set.

Proof

It is trivial.

Definition 2.5

An ideal topological space $(X,\tau,\mathcal{I})$ is said to be pre$-\mathcal{I}-T_1$ space if for each pair of distinct elements $x,y \in X$, there exist two pre$-\mathcal{I}-$open sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Theorem 2.6

An ideal topological space $(X,\tau,\mathcal{I})$ is pre$-\mathcal{I}-T_1$ space if and only if $\{x\}$ is a pre$-\mathcal{I}-$closed set for each $x \in X$.

Proof

Suppose X is a pre$-\mathcal{I}-T_1$ space. Let $x \in X$. To prove that $\{x\}$ is a pre$-\mathcal{I}-$closed, it is enough to prove that $\{x\}^c$ is a pre$-\mathcal{I}-$open set. If $\{x\}^c = \emptyset$, then it is clear. Suppose $\{x\}^c \neq \emptyset$. Let $y \in \{x\}^c$. Then $y \neq x$. Therefore, there exist two pre$-\mathcal{I}-$open sets U_x, V_y such that $x \notin V_y$ and $y \notin U_x$. Thus we get a family of pre$-\mathcal{I}-$open sets such that $x \notin V_y$ and $y \notin U_x$. Clearly $\{x\}^c = U_{y \neq x}\{V_y\}$. By Theorem 2.4, $\{x\}^c$ is a pre$-\mathcal{I}-$open set. Hence $\{x\}$ is pre$-\mathcal{I}-$closed. Conversely, suppose that $\{x\}$ is pre$-\mathcal{I}-$closed for each $x \in X$. Let $x,y \in X$ and $x \neq y$. By assumption, $\{x\}$ and $\{y\}$ are pre$-\mathcal{I}-$closed sets. This implies $\{x\}^c$ and $\{y\}^c$ are pre$-\mathcal{I}-$open sets in X. As $y \neq x$, we get that $x \in \{y\}^c$ and $y \in \{x\}^c$. Hence X is a pre$-\mathcal{I}-T_1$ space.

Definition 2.7

A subset A of X is called a pre$-\mathcal{I}-$neighbourhood of $x \in X$ if there exists, $U \in P\mathcal{I}O(X)$ such that $U \subset A$.

Theorem 2.8

Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. Then the following hold for the pre$-\mathcal{I}-$closure operator.

(a) A is pre$-\mathcal{I}-$closed if and only if $A = p\mathcal{I}cl(A)$.
(b) $p\mathcal{I}cl(A) \subset p\mathcal{I}cl(B)$ if $A \subset B$.
(c) $p\mathcal{I}cl(p\mathcal{I}cl(A)) = p\mathcal{I}cl(A)$.
(d) $p\mathcal{I}cl(A)$ is a pre$-\mathcal{I}-$closed set in X containing A.

Theorem 2.9

For every subset A of X, in an ideal topological space $(X,\tau,\mathcal{I})$ the following hold.

(a) $p\mathcal{I}cl(X-A) = X - p\mathcal{I}int(A)$.
(b) $p\mathcal{I}int(X-A) = X - p\mathcal{I}cl(A)$.

Proof

It is obvious.

Theorem 2.10

A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is pre$-\mathcal{I}-$open if and only if it is pre$-\mathcal{I}-$neighbourhood of each of its points.

Proof

Let $G \subset X$ be a pre$-\mathcal{I}-$open set. Then by definition, it is clear that G is a pre$-\mathcal{I}-$neighbourhood of each of its points.

Conversely, suppose G is a pre$-\mathcal{I}-$neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in P\mathcal{I}O(X)$ such that $S_x \subset G$. Clearly, $G = U\{S_x | x \in G\}$. Since each S_x is pre$-\mathcal{I}-$open, it follows that G is pre$-\mathcal{I}-$open by Theorem 2.4.

Definition 2.11

A point $x \in X$ is called a pre$-\mathcal{I}-$interior point of $A \subset X$ if there exists $G \in P\mathcal{I}O(X,\tau,\mathcal{I})$ such that $x \in G \subset A$, equivalently, $x \in p\mathcal{I}int(A)$.

Theorem 2.12

Let $(X,\tau,\mathcal{I})$ be an ideal topological space $A \subset X$, and $x \in X$. Then x is a pre$-\mathcal{I}-$interior point of A if and only if A is a pre$-\mathcal{I}-$neighbourhood of x.

Proof

It is obvious.

Remark 2.13 Since every open set is pre$-\mathcal{I}-$open, every interior point of A is a pre$-\mathcal{I}-$interior point of A. Thus, $int(A) \subset p\mathcal{I}int(A)$. In general, $int(A) \neq p\mathcal{I}int(A)$ is shown by the following Example 2.14.

Example 2.14

Consider the ideal space $(X,\tau,\mathcal{I})$ where $X = \{a,b,c\}$, $\tau = \{\emptyset,\{b\},X\}$ and $\mathcal{I} = \{\emptyset,\{a\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. If $A = \{a,b\}$, then $int(A) = \{b\}$ and $p\mathcal{I}int(A) = \{a,b\}$. This shows that $int(A) \neq p\mathcal{I}int(A)$.

Theorem 2.15

Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A \subset X$. Then $p\mathcal{I}int(A)$ is the largest pre$-\mathcal{I}-$open subset of X contained in A.

Proof

Assume that U is a pre$-\mathcal{I}-$open set with $U \subset A$. Let $x \in U$. Then, $x \in U \subset A$. Thus, A is a pre$-\mathcal{I}-$neighbourhood of $x \in U$. This shows that x is a pre$-\mathcal{I}-$interior point of A and so by Theorem 2.12, $x \in p\mathcal{I}int(A)$. Thus $x \in U$ implies that $x \in p\mathcal{I}int(A)$. Thus $U \subset p\mathcal{I}int(A)$. Therefore, $p\mathcal{I}int(A)$ contains every pre$-\mathcal{I}-$open set U contained in A and hence $p\mathcal{I}int(A)$ is the largest pre$-\mathcal{I}-$open set contained in A.

In view of Theorem 2.15, one can prove the following.

Theorem 2.16

A subset A of an ideal topological space is pre$-\mathcal{I}-$open if and only if $A = p\mathcal{I}int(A)$.

Theorem 2.17

Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. If $A \subset B$, then $p\mathcal{I}int(A) \subset p\mathcal{I}int(B)$.

Proof

If $A = \emptyset$, then $p\mathcal{I}int(A) = \emptyset$ and so $p\mathcal{I}int(A) = \emptyset \subset p\mathcal{I}int(B)$. Also, when $A \neq \emptyset$, $p\mathcal{I}int(A) = \emptyset$, then also the result is clear. Suppose $p\mathcal{I}int(A) \neq \emptyset$, then for each $x \in p\mathcal{I}int(A)$, there exists a pre$-\mathcal{I}-$open set G containing x and contained in A and therefore G is contained in B as $A \subset B$. Consequently, x is a pre$-\mathcal{I}-$interior point of B. Thus, each point of $p\mathcal{I}int(A)$ is also a point of $p\mathcal{I}int(B)$ whenever $A \subset B$. Therefore, $p\mathcal{I}int(A) \subset p\mathcal{I}int(B)$

whenever $A \subset B$.

Remark 2.18 $p\mathcal{I}int(A) = P\mathcal{I}int(B)$ does not imply that $A = B$. This is shown by the following Example 2.19.

Example 2.19

Consider the ideal space $(X,\tau,\mathcal{I})$ where $X = \{a,b,c\}, \tau = \{\emptyset,\{a\},\{c\},\{a,c\},\{b,c\},X\}$ and $\mathcal{I} = \{\emptyset,\{a,\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{a\},\{c\},\{a,c\},\{b,c\},X\}$. Take $A = \{a\}$, $B = \{a,b\}$. Then $p\mathcal{I}int(A) = p\mathcal{I}int(B)$ but $A \neq B$.

Theorem 2.20

Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. Then
(a) $p\mathcal{I}int(A) \cup p\mathcal{I}int(B) \subset p\mathcal{I}int(A \cup B)$.
(b) $p\mathcal{I}int(A \cap B) \subset p\mathcal{I}int(A) \cap p\mathcal{I}int(B)$.

Proof

The proof follows by Theorem 2.17. In general, $p\mathcal{I}int(A \cap B) \neq p\mathcal{I}int(A) \cap p\mathcal{I}int(B)$, as shown by the following Example 2.21.

Example 2.21

Consider the ideal space $(X,\tau,\mathcal{I})$ where $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a\},\{c\},\{a,c\},\{b,c\},X\}$ and $\mathcal{I} = \{\emptyset,\{d\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\},\{a,c,d\},X\}$. Take $A = \{b,c\}$ and $B = \{a,c,d\}$ then $A \cap B = \{c\}$. Then we have $p\mathcal{I}int(A) = \{b,c\}$, $p\mathcal{I}int(B) = \{a,c,d\}$ and $p\mathcal{I}int(A \cap B) = \emptyset$. Thus, it follows that $p\mathcal{I}int(A \cap B) \neq p\mathcal{I}int(A) \cap p\mathcal{I}int(B)$.

Theorem 2.22

Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. Then, $p\mathcal{I}int(A - B) \subset p\mathcal{I}int(A) - p\mathcal{I}int(B)$.

Proof

Let $x \in p\mathcal{I}int(A - B)$ which implies that there exists a pre$-\mathcal{I}-$open set U_x of x such that $U_x \subset (A - B) \subset A$. This shows that $U_x \cap B = \emptyset$ and hence $x \notin p\mathcal{I}int(B)$, $x \in p\mathcal{I}int(A)$. Therefore, $p\mathcal{I}int(A - B) \subset p\mathcal{I}int(A) - p\mathcal{I}int(B)$.

The equality of the above Theorem 2.22, does not hold in general, as illustrated by the following Example 2.23.

Example 2.23

Consider the ideal space $(X,\tau,\mathcal{I})$ where $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a\},\{b\},\{a,b,c\},X\}$ and $\mathcal{I} = \{\emptyset,\{d\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X\}$. Let $A = \{a,b,d\}$ and $B = \{a,b\}$. Then $p\mathcal{I}int(A) = \{a,b,d\}$, $p\mathcal{I}int(B) = \{a,b\}$. Therefore, $p\mathcal{I}int(A) - p\mathcal{I}int(B) = \{a,b,d\} - \{a,b\} = \{d\}$, $p\mathcal{I}int(A - B) = p\mathcal{I}int\{d\} = \emptyset$. Therefore, $p\mathcal{I}int(A - B) \neq p\mathcal{I}int(A) - p\mathcal{I}int(B)$.

3. PRE$-\mathcal{I}-$DERIVED SET

In this section, we introduced pre$-\mathcal{I}-$limit point of subset in ideal topological spaces.

Definition 3.1

Let $(X,\tau,\mathcal{I})$ be an ideal space. A point $x \in X$ is said to be a pre$-\mathcal{I}-$limit point of A if and only if for each $U \in P\mathcal{I}O(X)$, $U \cap (A - \{x\}) \neq \emptyset$.

Remark 3.2 Since every open set is pre$-\mathcal{I}-$open, it follows that every pre$-\mathcal{I}-$limit point A is a limit point of A.

Definition 3.3

The set of all pre$-\mathcal{I}-$limit point of A is said to be the pre$-\mathcal{I}-$derived set of A and is denoted by $p\mathcal{I}d(A)$.

Theorem 3.4

A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is pre$-\mathcal{I}-$closed if and only if it contains the set of all of its pre$-\mathcal{I}-$limit points.

Proof

Since A is pre$-\mathcal{I}-$closed, $X-A$ is pre$-\mathcal{I}-$open. Thus A is pre$-\mathcal{I}-$closed if and only if each point of $(X-A)$ has a pre$-\mathcal{I}-$neighbourhood contained in $(X-A)$ if and only if no point of $(X-A)$ is pre$-\mathcal{I}-$limit point of A (or) equivalently, A contains each of its pre$-\mathcal{I}-$limit points.

Theorem 3.5

Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. Then $A\subset B$ implies that $p\mathcal{I}d(A)\subset p\mathcal{I}d(B)$.

Proof

Let $x\in X$ be a pre$-\mathcal{I}-$limit point of A. Then by definition, for each $U\in P\mathcal{I}O(X)$, we have $U\cap(A-\{x\})\neq\emptyset$ and hence it follows that $U\cap(B-\{x\})\neq\emptyset$. Thus, x is a pre$-\mathcal{I}-$limit point of B. Hence $p\mathcal{I}d(A)\subset p\mathcal{I}d(B)$.

Theorem 3.6

Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. Then the following hold.
(a) $p\mathcal{I}d(\emptyset)=\emptyset$.
(b) If $x\in p\mathcal{I}d(A)$, then $x\in p\mathcal{I}d(A-\{x\})$.
(c) $p\mathcal{I}d(A)\cup p\mathcal{I}d(B)\subset p\mathcal{I}d(A\cup B)$.
(d) $p\mathcal{I}d(A\cap B)\subset p\mathcal{I}d(A)\cap p\mathcal{I}d(B)$.

Proof

(a) The proof is clear.
(b) If $x\in p\mathcal{I}d(A)$, then x is a pre$-\mathcal{I}-$limit point of A and so by definition, every pre$-\mathcal{I}-$neighbourhood of x contains at least one point of A other than x. Consequently, x is a pre$-\mathcal{I}-$limit point of $A-\{x\}$ and therefore $x\in p\mathcal{I}d(A-\{X\})$. Thus $X\in p\mathcal{I}d(A)$ implies $x\in p\mathcal{I}d(A-\{x\})$.
(c) As a consequence of Theorem 3.5, $A\subset A\cup B$ implies that $p\mathcal{I}d(A)\subset p\mathcal{I}d(A\cup B)$ and $B\subset A\cup B$ implies that $p\mathcal{I}d(B)\subset p\mathcal{I}d(A\cup B)$. These two together implies that $p\mathcal{I}d(A)\cup p\mathcal{I}d(B)\subset p\mathcal{I}d(A\cup B)$.
(d) Since $A\cap B\subset A$ and $A\cap B\subset B$, by Theorem 3.5, $p\mathcal{I}d(A\cap B)\subset p\mathcal{I}d(A)$ and $p\mathcal{I}d(A\cap B)\subset p\mathcal{I}d(B)$. Hence $p\mathcal{I}d(A\cap B)\subset p\mathcal{I}d(A)\cap p\mathcal{I}d(B)$.

Theorem 3.7

Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subset X$. Then, $A\cup p\mathcal{I}d(A)$ is pre$-\mathcal{I}-$closed.

Proof

To prove that $A\cup p\mathcal{I}d(A)$ is pre$-\mathcal{I}-$closed, it is enough to prove that $A\cup p\mathcal{I}d(A))^c$ is pre$-\mathcal{I}-$open. If $(A\cup p\mathcal{I}d(A))^c=\emptyset$, then it is clearly a pre$-\mathcal{I}-$open set. Suppose $(A\cup p\mathcal{I}d(A))^c\neq\emptyset$. Let x be an arbitrary point of $(A\cup p\mathcal{I}d(A))^c$ so that $x\in A^c$ and $x\notin p\mathcal{I}d(A)$. Now $x\notin p\mathcal{I}d(A)$ implies that x is not a pre$-\mathcal{I}-$limit point of A and so there exists a pre$-\mathcal{I}-$open set N_x containing the point x and not containing any point of A other than x. Since $x\notin A$, N_x contains no point of A which implies that $N_x\subset(X-A)$. Again, N_x is a pre$-\mathcal{I}-$neighbourhood of each of its points. But N_x does not contain any point of A. Therefore, no point of N_x can be a pre$-\mathcal{I}-$limit point of A. Therefore, no point of N_x belongs to $p\mathcal{I}d(A)$. This implies that $N_x\subset(X-(A\cup p\mathcal{I}d(A)))$. Hence it follows that $x\in N_x\subset(X-(A\cup p\mathcal{I}d(A)))$ which implies that $X-(A\cup p\mathcal{I}d(A))$ is pre$-\mathcal{I}-$open. Hence $A\cup p\mathcal{I}d(A)$ is pre$-\mathcal{I}-$closed.

Theorem 3.8

Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subset X$. Then $p\mathcal{I}cl(A)=A\cup p\mathcal{I}d(A)$.

Proof

By Theorem 3.7, $A\cup p\mathcal{I}d(A)$ is a pre$-\mathcal{I}-$closed set. Also as $A\subset A\cup p\mathcal{I}d(A)$, $A\cup p\mathcal{I}d(A)$ is a pre$-\mathcal{I}-$closed set containing A. Now $p\mathcal{I}cl(A)$ is the smallest pre$-\mathcal{I}-$closed set containing A. Therefore, $p\mathcal{I}cl(A)\subset A\cup p\mathcal{I}d(A)$.

Again $A \subset p\mathcal{I}cl(A)$ implies that $p\mathcal{I}d(A) \subset p\mathcal{I}d(p\mathcal{I}cl(A)) \subset p\mathcal{I}cl(A)$. Thus $A \subset p\mathcal{I}cl(A)$ and $p\mathcal{I}d(A) \subset p\mathcal{I}cl(A)$ which implies that $A \cup p\mathcal{I}d(A) \subset p\mathcal{I}cl(A)$. Hence $p\mathcal{I}cl(A) = A \cup p\mathcal{I}d(A)$.

It is obvious that $p\mathcal{I}cl(A) \subset cl(A)$. The converse is false as shown by the following Example 3.9.

Example 3.9

Consider the ideal space $(X,\tau,\mathcal{I})$ where $X = \{a,b,c\}$, $\tau = \{\emptyset,\{b,c\},X\}$ and $\mathcal{I} = \{\emptyset,\{a\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$. If $A = \{b\}$, then $cl(A) = X$ and $p\mathcal{I}cl(A) = \{b\}$. Hence $cl(A) \nsubseteq p\mathcal{I}cl(A)$.

Remark 3.10 If $p\mathcal{I}cl(A) = p\mathcal{I}cl(B)$., it does not imply that $A = B$. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{b\},\{d\},\{b,d\},X\}$ and $\mathcal{I} = \{\emptyset,\{a\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{b\},\{d\},\{c,d\},\{a,b,d\},\{b,c,d\},X\}$. Take $A = \{d\}, B = \{a,d\}$. Then $p\mathcal{I}cl(A) = p\mathcal{I}cl(B) = \{a,c,d\}$.

Theorem 3.11

Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. Then the following hold.
(a) $p\mathcal{I}cl(A) \cup p\mathcal{I}cl(B) \subset p\mathcal{I}cl(A \cup B)$.
(b) $p\mathcal{I}cl(A \cap B) \subset p\mathcal{I}cl(A) \cap p\mathcal{I}cl(B)$.

Proof

(a) Now $A \subset (A \cup B)$ implies $p\mathcal{I}cl(A) \subset p\mathcal{I}cl(A \cup B)$. Also $B \subset (A \cup B)$ implies $p\mathcal{I}cl(B) \subset p\mathcal{I}cl(A \cup B)$. Thus, $p\mathcal{I}cl(A) \cup p\mathcal{I}cl(B) \subset P\mathcal{I}cl(A \cup B)$.
(b) Since $A \cap B \subset A$ and $A \cap B \subset B$, $p\mathcal{I}cl(A \cap B) \subset p\mathcal{I}cl(A)$ and $p\mathcal{I}cl(A \cap B) \subset p\mathcal{I}cl(B)$. Hence $p\mathcal{I}cl(A \cap B) \subset p\mathcal{I}cl(A) \cap p\mathcal{I}cl(B)$.

4. PRE$-\mathcal{I}-$FRONTIER OF A SET

Definition 4.1

Let A be a subset of an ideal space $(X,\tau,\mathcal{I})$. $p\mathcal{I}cl(A) - p\mathcal{I}int(A)$ is said to be the pre$-\mathcal{I}-$frontier of $A \subset X$ and is denoted by $p\mathcal{I}F_r(A)$.

It is obvious that $p\mathcal{I}F_r(A) \subset F_r(A)$, the frontier of A. But the converse need not be true as shown by the following Example 4.2.

Example 4.2

Consider the ideal space $(X,\tau,\mathcal{I})$ where $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a,b\},\{a,b,c\},X\}$ and $\mathcal{I} = \{\emptyset,\{d\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X\}$. Let $A = \{a,b,d\}$. Then $int(A) = \{a,b\}$, $cl(A) = X$, $p\mathcal{I}cl(A) = X$, $p\mathcal{I}int(A) = \{a,b,d\}$, $p\mathcal{I}F_r(A) = \{c\}$ and $F_r(A) = \{c,d\}$. This shows that $F_r(A) \nsubseteq p\mathcal{I}F_r(A)$.

Theorem 4.3

Let A be a subset of an ideal topological space $(X,\tau,\mathcal{I})$. Then the following hold.
(a) $p\mathcal{I}cl(A) = p\mathcal{I}int(A) \cup p\mathcal{I}F_r(A)$.
(b) $p\mathcal{I}int(A) \cap p\mathcal{I}F_r(A) = \emptyset$.
(c) $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) \cap p\mathcal{I}cl(X - A)$.

Proof

(a) $p\mathcal{I}int(A) \cup p\mathcal{I}F_r(A) = p\mathcal{I}int(A) \cup (p\mathcal{I}cl(A) - p\mathcal{I}int(A)) = p\mathcal{I}cl(A)$, by Definition 4.1.
(b) $p\mathcal{I}int(A) \cap p\mathcal{I}F_r(A) = p\mathcal{I}int(A) \cap (p\mathcal{I}cl(A) - p\mathcal{I}int(A)) = p\mathcal{I}int(A) \cap p\mathcal{I}cl(A) - (p\mathcal{I}int(A) \cap p\mathcal{I}int(A)) = p\mathcal{I}int(A) - p\mathcal{I}int(A) = \emptyset$.
(c) $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}cl(A) \cap (X - p\mathcal{I}int(A)) = p\mathcal{I}cl(A) \cap p\mathcal{I}cl(X - A)$.

Definition 4.4

A subset A of a space $(X,\tau,\mathcal{I})$ with an ideal $\mathcal{I}$ is said to be pre$-\mathcal{I}-$regular if it is pre$-\mathcal{I}-$open and pre$-\mathcal{I}-$closed.

Theorem 4.5

Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A \subset X$. Then $p\mathcal{I}F_r(A) = \emptyset$ if and only if A is pre$-\mathcal{I}-$regular.

Proof

The proof is clear.

Theorem 4.6

Let A be a subset of an ideal topological space $(X,\tau,\mathcal{I})$. Then the following hold.
(a) $p\mathcal{I}F_r(A) = p\mathcal{I}fr(X-A)$.
(b) $A \in P\mathcal{I}O(X,\tau,\mathcal{I})$ if and only if $p\mathcal{I}F_r(A) \subset (X-A)$ (that is $A \cap p\mathcal{I}F_r(A) = \emptyset$.)
(c) $A \in P\mathcal{I}C(X,\tau,\mathcal{I})$ if and only if $p\mathcal{I}F_r(A) \subset A$.

Proof

(a) The proof is clear.
(b) Let $A \in P\mathcal{I}O(X,\tau,\mathcal{I})$. Then by definition, $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}cl(A) - A$. Therefore, $A \cap p\mathcal{I}F_r(A) = A \cap (p\mathcal{I}cl(A) - A) = \emptyset$. Conversely, if $A \cap p\mathcal{I}F_r(A) = \emptyset$ implies that $(p\mathcal{I}cl(A) - p\mathcal{I}int(A)) \cap A = \emptyset$ which implies $(A \cap p\mathcal{I}cl(A)) \cap (X - p\mathcal{I}int(A)) = \emptyset$. Hence $A \cap (X - p\mathcal{I}int(A)) = \emptyset$ as $A \subset p\mathcal{I}cl(A)$. Thus, $A \subset X - (X - p\mathcal{I}int(A))$ implies that $A \subset p\mathcal{I}int(A)$. But on the otherhand, $p\mathcal{I}int(A) \subset A$. It follows that $A \in P\mathcal{I}O(X,\tau,\mathcal{I})$.
(c) The proof follows from (a) and (b).

Remark 4.7 Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. Then $A \subset B$ does not imply that either $p\mathcal{I}F_r(A) \subset p\mathcal{I}F_r(B)$ (or) $p\mathcal{I}F_r(B) \subset p\mathcal{I}F_r(A)$. This can be verified by the following Example 4.8.

Example 4.8

Consider the ideal space $(X,\tau,\mathcal{I})$ where $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a,b\},\{a,b,c\},X\}$ and $\mathcal{I} = \{\emptyset,\{d\}\}$. Then $P\mathcal{I}O(X,\tau,\mathcal{I}) = \{\emptyset,\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X\}$.

Case (i)

Let $A = \{d\}$, $B = \{a,c,d\}$. Then $A \subset B$. Also, $p\mathcal{I}F_r(A) = \{d\}$ and $p\mathcal{I}F_r(B) = \{b\}$. Hence $p\mathcal{I}F_r(A) \nsubseteq p\mathcal{I}F_r(B)$.

Case (ii)

Let $A = \{d\}$, $B = \{c,d\}$ then $A \subset B$. Also, $p\mathcal{I}F_r(A) = \{d\}$ and $p\mathcal{I}F_r(B) = P\mathcal{I}cl(B) - p\mathcal{I}int(B) = \{c,d\} - \emptyset = \{c,d\}$. Hence $p\mathcal{I}F_r(B) \nsubseteq p\mathcal{I}F_r(A)$.

Theorem 4.9

For any subset A of an ideal topological space $(X,\tau,\mathcal{I})$, we have $p\mathcal{I}F_r(p\mathcal{I}F_r(A)) \subset p\mathcal{I}F_r(A)$.

Proof

By Theorem 4.3(c), $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) \cap p\mathcal{I}cl(X-A)$. Therefore $p\mathcal{I}F_r(p\mathcal{I}F_r(A)) = P\mathcal{I}cl(p\mathcal{I}F_r(A)) \cap p\mathcal{I}cl(X - p\mathcal{I}F_r(A)) \subset p\mathcal{I}cl(p\mathcal{I}F_r(A)) = p\mathcal{I}F_r(A)$, since $p\mathcal{I}F_r(A)$ is pre$-\mathcal{I}-$closed.

Theorem 4.10

Let A be a subset of an ideal topological space $(X,\tau,\mathcal{I})$. Then the following hold.
(a) $p\mathcal{I}F_r(p\mathcal{I}int(A)) \subset p\mathcal{I}F_r(A)$.
(b) $p\mathcal{I}F_r(P\mathcal{I}cl(A)) \subset p\mathcal{I}F_r(A)$.

Proof

(a) $p\mathcal{I}F_r(p\mathcal{I}int(A)) = p\mathcal{I}cl(p\mathcal{I}int(A)) - p\mathcal{I}int(p\mathcal{I}int(A)) = p\mathcal{I}cl(p\mathcal{I}int(A)) - p\mathcal{I}int(A) \subset p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}F_r(A)$.

(b) $p\mathcal{I}F_r(p\mathcal{I}cl(A)) = p\mathcal{I}cl(p\mathcal{I}cl(A)) - p\mathcal{I}int(p\mathcal{I}cl(A)) = p\mathcal{I}cl(A) - p\mathcal{I}int(p\mathcal{I}cl(A)) \subset p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}F_r(A)$.

Theorem 4.11

A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is pre$-\mathcal{I}-$open if and only if $p\mathcal{I}F_r(A) = p\mathcal{I}d(A)$.

Proof

Let A be a pre$-\mathcal{I}-$open set then $p\mathcal{I}int(A) = A$. Now, $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}cl(A) - A$. Since $p\mathcal{I}cl(A) = A \cup p\mathcal{I}d(A)$, $p\mathcal{I}F_r(A) = (A \cup p\mathcal{I}d(A)) - A = p\mathcal{I}d(A)$.

Conversely, let $p\mathcal{I}F_r(A) = p\mathcal{I}d(A)$. Then $p\mathcal{I}d(A) = p\mathcal{I}cl(A) - p\mathcal{I}int(A) = (A \cup p\mathcal{I}d(A)) - p\mathcal{I}int(A)$. Hence $A - p\mathcal{I}int(A) = \emptyset$. Therefore, A is pre$-\mathcal{I}-$open.

Theorem 4.12

Let $\{A_\alpha | \alpha \in I\}$ be any family of subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If $\cup_{\alpha\in I} p\mathcal{I}cl(A_\alpha)$ is pre$-\mathcal{I}-$closed, then $\cup_{\alpha\in I} p\mathcal{I}cl(A_\alpha) = p\mathcal{I}cl(\cup_{\alpha\in I} A_\alpha)$.

Proof

Since $A_\alpha \subset \cup_{\alpha\in I}$, $p\mathcal{I}cl(A_\alpha) \subset p\mathcal{I}cl(\cup_{\alpha\in I})$ and hence $\cup_{\alpha\in I} p\mathcal{I}cl(A_\alpha) \subset p\mathcal{I}cl(\cup_{\alpha\in I} A_\alpha)$. We will show that $p\mathcal{I}cl(\cup_{\alpha\in I} A_\alpha) \subset \cup_{\alpha\in I} p\mathcal{I}cl(A_\alpha)$. Let $x \in p\mathcal{I}cl(\cup_{\alpha\in I} A_\alpha)$. Since $\cup_{\alpha\in I} p\mathcal{I}cl(A_\alpha)$ is pre$-\mathcal{I}-$closed it contains all its pre$-\mathcal{I}-$limit points of $\cup_{\alpha\in I} p\mathcal{I}cl(A_\alpha)$ and so, there exists a pre$-\mathcal{I}-$neighbourhood U_x of x such that $U_x \cap (\cup_{\alpha\in I} p\mathcal{I}cl(A_\alpha)) = \emptyset$. This implies that $U_x \cap p\mathcal{I}cl(A_\alpha) = \emptyset$ for every $\alpha \in I$ and hence $U_x \cap A_\alpha = \emptyset$ which is a contradiction to $x \in p\mathcal{I}cl(\cup_{\alpha\in I} A_\alpha)$. Therefore, $p\mathcal{I}cl(\cup_{\alpha\in I} A_\alpha) \subset \cup_{\alpha\in I} p\mathcal{I}cl(A_\alpha)$ which completes the proof.

REFERENCES

[1] J.Dontchev, On Pre$-\mathcal{I}-$Open Sets and a Decomposition of $\mathcal{I}-$Continuity, Banyan Mathematical Journal, Vol. 2, No.1, 1996, pp. 5.

[2] Julian Dontchev, Maximilian Ganster and David Rose, Ideal Resolvability, Topology and its Applications, Vol. 93, No. 1, 1999, pp. 1-16.

[3] D.Jankovic and T.R.Hamlett, Compatible Extensions of Ideals, Boll. UMI, Vol. 7, 1992, pp. 453–465.

[4] Kasimierz Kuratowski, Topology, Vol. 1, Academic Press, New York, 1966, pp. 1-840.

[5] A.S.Mashhour, M.E.Abd El-Monsef and S.N.El-Deeb, On Precontinuous and Weak Precontinuous Mappings, Proceedings of the Mathematical and Physical Society of Egypt, Vol. 53, No. 1982, 1982, pp. 47-53.

[6] G.B.Navalagi, Pre-neighbourhoods, The Mathematics Education, Vol. 32, No. 4, 1998, pp. 201-206.

[7] R.Vaidyanathaswamy, The Localization Theory in Set Topology, Proceedings of the Indian Academy of Sciences: Mathematical Sciences, Vol. 20, 1945, pp. 51-61.

[8] Jess Mathew, Subdivided Stars Super (b, d) – Edge Antimagic Total Graph Labelling, DJ Journal of Engineering and Applied Mathematics, Vol. 1, No.1, 2015, pp. 30-34, http://dx.doi.org/10.18831/djmaths.org/2015011005.

[9] R.Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, New York, 1960, pp. 1-148.