

RESEARCH ARTICLE

Properties of pre- \mathcal{I} -open sets **R Sajuntha¹¹Department of Mathematics, Ponjesly College of Engineering, Nagercoil- 629 003, Tamil Nadu, India.

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ABSTRACT

In 1982, [5] have defined the notion of preopen sets. Quite recently in 2002, [6] has defined pre neighbourhoods, pre-interior point, prelimit point, prederived set and prefrontier of set. For these sets we define the notion of pre- \mathcal{I} -interior point, pre- \mathcal{I} -limit point, and pre- \mathcal{I} -frontier of a set using pre- \mathcal{I} -open sets and pre- \mathcal{I} -closed sets in ideal topological space. We also obtained a necessary and sufficient condition for an ideal topological space to be a pre- \mathcal{I} - T_1 space.

Keywords: Ideal spaces, Pre- \mathcal{I} -open sets, Pre- \mathcal{I} -closed sets, Pre- \mathcal{I} - T_1 space, Pre- \mathcal{I} -interior point, Pre- \mathcal{I} -frontier of a set.

1. INTRODUCTION AND PRELIMINARIES

The subject of ideals in topological space has been studied by [4] and [7-9]. An ideal \mathcal{I} on a topological space (X, τ) is a non empty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal \mathcal{I} on X and if $\wp(X)$ is the set of all subsets of X , a set operator $(\cdot)^* : \wp(X) \rightarrow \wp(X)$, called a local function [4] of A with respect to τ and \mathcal{I} , is defined as follows: for $A \subset X, A^*(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}, \text{ for every } \{U \in \tau(x)\} \text{ where } \tau(x) = \{U \in \tau | x \in U\}\}$. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(X, \tau)$, called the \star -topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [7, 8]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. If \mathcal{I} is an ideal on X , then (X, τ, \mathcal{I}) is called an ideal space. A subset A of an ideal space (X, τ, \mathcal{I}) is said to be \mathcal{I} -open [3] if $A \subset int(A^*)$. A subset A of an ideal space (X, τ, \mathcal{I}) is said to be pre- \mathcal{I} -open [1] if $A \subset int(cl^*(A))$. The complement of pre- \mathcal{I} -open set is called pre- \mathcal{I} -closed. The family of all pre- \mathcal{I} -open sets in (X, τ, \mathcal{I}) is denoted by $P\mathcal{I}O(X, \tau, \mathcal{I})$ or simply $P\mathcal{I}O(X)$. Clearly $\tau \subset P\mathcal{I}O(X)$. The largest pre- \mathcal{I} -open set contained in A , denoted by $p\mathcal{I}int(A)$, is called the pre- \mathcal{I} -interior of A . The smallest pre- \mathcal{I} -closed set containing A , denoted by $p\mathcal{I}cl(A)$, is called the pre- \mathcal{I} -closure of A . A subset A of an ideal space (X, τ, \mathcal{I}) is \star -dense [2] if $cl^*(A) = X$.

2. SOME MORE RESULTS ON PRE- \mathcal{I} -OPEN SETS**Theorem 2.1**

If $A \subset X$ is \star -dense in (X, τ, \mathcal{I}) , then A is a pre- \mathcal{I} -open set.

Proof

Suppose A is \star -dense in X . Then $cl^*(A) = X$ which implies that $int(cl^*(A)) = int(X) = X$. Therefore, $A \subset int(cl^*(A))$. Hence A is pre- \mathcal{I} -open.

The converse of the Theorem 2.1 need not be true as shown by the following Example 2.2.

Example 2.2

Consider the ideal space (X, τ, \mathcal{I}) where $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $A = \{b\}$. Then $cl^*(A) = \{b, c\}$ and $int(cl^*\{b\}) = b$. Therefore, A is pre- \mathcal{I} -open. But $cl^*(\{b\}) = \{b, c\} \neq X$. Hence A is not \star -dense.

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Theorem 2.3

Let A be a subset of an open set B in an ideal topological space (X, τ, \mathcal{I}) such that $A \subset B \subset cl^*(A)$. Then A is a pre- \mathcal{I} -open set.

Proof

Given that B is an open set in X so that, $B = int(B)$. Now $A \subset B \subset cl^*(A)$ implies that $int(A) \subset int(B) \subset int(cl^*(A))$ which inturn implies $int(A) \subset B \subset int(cl^*(A))$. Therefore, $A \subset int(cl^*(A))$. Hence A is pre- \mathcal{I} -open in X .

Theorem 2.4

In an ideal topological space (X, τ, \mathcal{I}) , arbitrary union of pre- \mathcal{I} -open sets is a pre- \mathcal{I} -open set.

Proof

It is trivial.

Definition 2.5

An ideal topological space (X, τ, \mathcal{I}) is said to be pre- $\mathcal{I} - T_1$ space if for each pair of distinct elements $x, y \in X$, there exist two pre- \mathcal{I} -open sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Theorem 2.6

An ideal topological space (X, τ, \mathcal{I}) is pre- $\mathcal{I} - T_1$ space if and only if $\{x\}$ is a pre- \mathcal{I} -closed set for each $x \in X$.

Proof

Suppose X is a pre- $\mathcal{I} - T_1$ space. Let $x \in X$. To prove that $\{x\}$ is a pre- \mathcal{I} -closed, it is enough to prove that $\{x\}^c$ is a pre- \mathcal{I} -open set. If $\{x\}^c = \emptyset$, then it is clear. Suppose $\{x\}^c \neq \emptyset$. Let $y \in \{x\}^c$. Then $y \neq x$. Therefore, there exist two pre- \mathcal{I} -open sets U_x, V_y such that $x \notin V_y$ and $y \notin U_x$. Thus we get a family of pre- \mathcal{I} -open sets such that $x \notin V_y$ and $y \notin U_x$. Clearly $\{x\}^c = \bigcup_{y \neq x} \{V_y\}$. By Theorem 2.4, $\{x\}^c$ is a pre- \mathcal{I} -open set. Hence $\{x\}$ is pre- \mathcal{I} -closed. Conversely, suppose that $\{x\}$ is pre- \mathcal{I} -closed for each $x \in X$. Let $x, y \in X$ and $x \neq y$. By assumption, $\{x\}$ and $\{y\}$ are pre- \mathcal{I} -closed sets. This implies $\{x\}^c$ and $\{y\}^c$ are pre- \mathcal{I} -open sets in X . As $y \neq x$, we get that $x \in \{y\}^c$ and $y \in \{x\}^c$. Hence X is a pre- $\mathcal{I} - T_1$ space.

Definition 2.7

A subset A of X is called a pre- \mathcal{I} -neighbourhood of $x \in X$ if there exists, $U \in \mathcal{PTO}(X)$ such that $U \subset A$.

Theorem 2.8

Let A and B be subsets of an ideal topological space (X, τ, \mathcal{I}) . Then the following hold for the pre- \mathcal{I} -closure operator.

- (a) A is pre- \mathcal{I} -closed if and only if $A = p\mathcal{I}cl(A)$.
- (b) $p\mathcal{I}cl(A) \subset p\mathcal{I}cl(B)$ if $A \subset B$.
- (c) $p\mathcal{I}cl(p\mathcal{I}cl(A)) = p\mathcal{I}cl(A)$.
- (d) $p\mathcal{I}cl(A)$ is a pre- \mathcal{I} -closed set in X containing A .

Theorem 2.9

For every subset A of X , in an ideal topological space (X, τ, \mathcal{I}) the following hold.

- (a) $p\mathcal{I}cl(X - A) = X - p\mathcal{I}int(A)$.
- (b) $p\mathcal{I}int(X - A) = X - p\mathcal{I}cl(A)$.

Proof

It is obvious.

Theorem 2.10

A subset A of an ideal topological space (X, τ, \mathcal{I}) is pre- \mathcal{I} -open if and only if it is pre- \mathcal{I} -neighbourhood of each of its points.

Proof

Let $G \subset X$ be a pre- \mathcal{I} -open set. Then by definition, it is clear that G is a pre- \mathcal{I} -neighbourhood of each of its points.

Conversely, suppose G is a pre- \mathcal{I} -neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in P\mathcal{I}O(X)$ such that $S_x \subset G$. Clearly, $G = U\{S_x | x \in G\}$. Since each S_x is pre- \mathcal{I} -open, it follows that G is pre- \mathcal{I} -open by Theorem 2.4.

Definition 2.11

A point $x \in X$ is called a pre- \mathcal{I} -interior point of $A \subset X$ if there exists $G \in P\mathcal{I}O(X, \tau, \mathcal{I})$ such that $x \in G \subset A$, equivalently, $x \in p\mathcal{I}int(A)$.

Theorem 2.12

Let (X, τ, \mathcal{I}) be an ideal topological space $A \subset X$, and $x \in X$. Then x is a pre- \mathcal{I} -interior point of A if and only if A is a pre- \mathcal{I} -neighbourhood of x .

Proof

It is obvious.

Remark 2.13 Since every open set is pre- \mathcal{I} -open, every interior point of A is a pre- \mathcal{I} -interior point of A . Thus, $int(A) \subset p\mathcal{I}int(A)$. In general, $int(A) \neq p\mathcal{I}int(A)$ is shown by the following Example 2.14.

Example 2.14

Consider the ideal space (X, τ, \mathcal{I}) where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. If $A = \{a, b\}$, then $int(A) = \{b\}$ and $p\mathcal{I}int(A) = \{a, b\}$. This shows that $int(A) \neq p\mathcal{I}int(A)$.

Theorem 2.15

Let (X, τ, \mathcal{I}) be an ideal topological space and $A \subset X$. Then $p\mathcal{I}int(A)$ is the largest pre- \mathcal{I} -open subset of X contained in A .

Proof

Assume that U is a pre- \mathcal{I} -open set with $U \subset A$. Let $x \in U$. Then, $x \in U \subset A$. Thus, A is a pre- \mathcal{I} -neighbourhood of $x \in U$. This shows that x is a pre- \mathcal{I} -interior point of A and so by Theorem 2.12, $x \in p\mathcal{I}int(A)$. Thus $x \in U$ implies that $x \in p\mathcal{I}int(A)$. Thus $U \subset p\mathcal{I}int(A)$. Therefore, $p\mathcal{I}int(A)$ contains every pre- \mathcal{I} -open set U contained in A and hence $p\mathcal{I}int(A)$ is the largest pre- \mathcal{I} -open set contained in A .

In view of Theorem 2.15, one can prove the following.

Theorem 2.16

A subset A of an ideal topological space is pre- \mathcal{I} -open if and only if $A = p\mathcal{I}int(A)$.

Theorem 2.17

Let A and B be subsets of an ideal topological space (X, τ, \mathcal{I}) . If $A \subset B$, then $p\mathcal{I}int(A) \subset p\mathcal{I}int(B)$.

Proof

If $A = \emptyset$, then $p\mathcal{I}int(A) = \emptyset$ and so $p\mathcal{I}int(A) = \emptyset \subset p\mathcal{I}int(B)$. Also, when $A \neq \emptyset$, $p\mathcal{I}int(A) = \emptyset$, then also the result is clear. Suppose $p\mathcal{I}int(A) \neq \emptyset$, then for each $x \in p\mathcal{I}int(A)$, there exists a pre- \mathcal{I} -open set G containing x and contained in A and therefore G is contained in B as $A \subset B$. Consequently, x is a pre- \mathcal{I} -interior point of B . Thus, each point of $p\mathcal{I}int(A)$ is also a point of $p\mathcal{I}int(B)$ whenever $A \subset B$. Therefore, $p\mathcal{I}int(A) \subset p\mathcal{I}int(B)$.

whenever $A \subset B$.

Remark 2.18 $p\mathcal{I}int(A) = p\mathcal{I}int(B)$ does not imply that $A = B$. This is shown by the following Example 2.19.

Example 2.19

Consider the ideal space (X, τ, \mathcal{I}) where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Take $A = \{a\}$, $B = \{a, b\}$. Then $p\mathcal{I}int(A) = p\mathcal{I}int(B)$ but $A \neq B$.

Theorem 2.20

Let A and B be subsets of an ideal topological space (X, τ, \mathcal{I}) . Then

- (a) $p\mathcal{I}int(A) \cup p\mathcal{I}int(B) \subset p\mathcal{I}int(A \cup B)$.
- (b) $p\mathcal{I}int(A \cap B) \subset p\mathcal{I}int(A) \cap p\mathcal{I}int(B)$.

Proof

The proof follows by Theorem 2.17. In general, $p\mathcal{I}int(A \cap B) \neq p\mathcal{I}int(A) \cap p\mathcal{I}int(B)$, as shown by the following Example 2.21.

Example 2.21

Consider the ideal space (X, τ, \mathcal{I}) where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}, X\}$. Take $A = \{b, c\}$ and $B = \{a, c, d\}$ then $A \cap B = \{c\}$. Then we have $p\mathcal{I}int(A) = \{b, c\}$, $p\mathcal{I}int(B) = \{a, c, d\}$ and $p\mathcal{I}int(A \cap B) = \emptyset$. Thus, it follows that $p\mathcal{I}int(A \cap B) \neq p\mathcal{I}int(A) \cap p\mathcal{I}int(B)$.

Theorem 2.22

Let A and B be subsets of an ideal topological space (X, τ, \mathcal{I}) . Then, $p\mathcal{I}int(A - B) \subset p\mathcal{I}int(A) - p\mathcal{I}int(B)$.

Proof

Let $x \in p\mathcal{I}int(A - B)$ which implies that there exists a pre- \mathcal{I} -open set U_x of x such that $U_x \subset (A - B) \subset A$. This shows that $U_x \cap B = \emptyset$ and hence $x \notin p\mathcal{I}int(B)$, $x \in p\mathcal{I}int(A)$. Therefore, $p\mathcal{I}int(A - B) \subset p\mathcal{I}int(A) - p\mathcal{I}int(B)$.

The equality of the above Theorem 2.22, does not hold in general, as illustrated by the following Example 2.23.

Example 2.23

Consider the ideal space (X, τ, \mathcal{I}) where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$. Let $A = \{a, b, d\}$ and $B = \{a, b\}$. Then $p\mathcal{I}int(A) = \{a, b, d\}$, $p\mathcal{I}int(B) = \{a, b\}$. Therefore, $p\mathcal{I}int(A) - p\mathcal{I}int(B) = \{a, b, d\} - \{a, b\} = \{d\}$, $p\mathcal{I}int(A - B) = p\mathcal{I}int\{d\} = \emptyset$. Therefore, $p\mathcal{I}int(A - B) \neq p\mathcal{I}int(A) - p\mathcal{I}int(B)$.

3. PRE- \mathcal{I} -DERIVED SET

In this section, we introduced pre- \mathcal{I} -limit point of subset in ideal topological spaces.

Definition 3.1

Let (X, τ, \mathcal{I}) be an ideal space. A point $x \in X$ is said to be a pre- \mathcal{I} -limit point of A if and only if for each $U \in P\mathcal{I}O(X)$, $U \cap (A - \{x\}) \neq \emptyset$.

Remark 3.2 Since every open set is pre- \mathcal{I} -open, it follows that every pre- \mathcal{I} -limit point A is a limit point of A .

Definition 3.3

The set of all pre- \mathcal{I} -limit point of A is said to be the pre- \mathcal{I} -derived set of A and is denoted by $p\mathcal{I}d(A)$.

Theorem 3.4

A subset A of an ideal topological space (X, τ, \mathcal{I}) is pre- \mathcal{I} -closed if and only if it contains the set of all of its pre- \mathcal{I} -limit points.

Proof

Since A is pre- \mathcal{I} -closed, $X - A$ is pre- \mathcal{I} -open. Thus A is pre- \mathcal{I} -closed if and only if each point of $(X - A)$ has a pre- \mathcal{I} -neighbourhood contained in $(X - A)$ if and only if no point of $(X - A)$ is pre- \mathcal{I} -limit point of A (or) equivalently, A contains each of its pre- \mathcal{I} -limit points.

Theorem 3.5

Let A and B be subsets of an ideal topological space (X, τ, \mathcal{I}) . Then $A \subset B$ implies that $p\mathcal{I}d(A) \subset p\mathcal{I}d(B)$.

Proof

Let $x \in X$ be a pre- \mathcal{I} -limit point of A . Then by definition, for each $U \in P\mathcal{I}O(X)$, we have $U \cap (A - \{x\}) \neq \emptyset$ and hence it follows that $U \cap (B - \{x\}) \neq \emptyset$. Thus, x is a pre- \mathcal{I} -limit point of B . Hence $p\mathcal{I}d(A) \subset p\mathcal{I}d(B)$.

Theorem 3.6

Let A and B be subsets of an ideal topological space (X, τ, \mathcal{I}) . Then the following hold.

- (a) $p\mathcal{I}d(\emptyset) = \emptyset$.
- (b) If $x \in p\mathcal{I}d(A)$, then $x \in p\mathcal{I}d(A - \{x\})$.
- (c) $p\mathcal{I}d(A) \cup p\mathcal{I}d(B) \subset p\mathcal{I}d(A \cup B)$.
- (d) $p\mathcal{I}d(A \cap B) \subset p\mathcal{I}d(A) \cap p\mathcal{I}d(B)$.

Proof

- (a) The proof is clear.
- (b) If $x \in p\mathcal{I}d(A)$, then x is a pre- \mathcal{I} -limit point of A and so by definition, every pre- \mathcal{I} -neighbourhood of x contains at least one point of A other than x . Consequently, x is a pre- \mathcal{I} -limit point of $A - \{x\}$ and therefore $x \in p\mathcal{I}d(A - \{x\})$. Thus $x \in p\mathcal{I}d(A)$ implies $x \in p\mathcal{I}d(A - \{x\})$.
- (c) As a consequence of Theorem 3.5, $A \subset A \cup B$ implies that $p\mathcal{I}d(A) \subset p\mathcal{I}d(A \cup B)$ and $B \subset A \cup B$ implies that $p\mathcal{I}d(B) \subset p\mathcal{I}d(A \cup B)$. These two together implies that $p\mathcal{I}d(A) \cup p\mathcal{I}d(B) \subset p\mathcal{I}d(A \cup B)$.
- (d) Since $A \cap B \subset A$ and $A \cap B \subset B$, by Theorem 3.5, $p\mathcal{I}d(A \cap B) \subset p\mathcal{I}d(A)$ and $p\mathcal{I}d(A \cap B) \subset p\mathcal{I}d(B)$. Hence $p\mathcal{I}d(A \cap B) \subset p\mathcal{I}d(A) \cap p\mathcal{I}d(B)$.

Theorem 3.7

Let (X, τ, \mathcal{I}) be an ideal space and $A \subset X$. Then, $A \cup p\mathcal{I}d(A)$ is pre- \mathcal{I} -closed.

Proof

To prove that $A \cup p\mathcal{I}d(A)$ is pre- \mathcal{I} -closed, it is enough to prove that $(A \cup p\mathcal{I}d(A))^c$ is pre- \mathcal{I} -open. If $(A \cup p\mathcal{I}d(A))^c = \emptyset$, then it is clearly a pre- \mathcal{I} -open set. Suppose $(A \cup p\mathcal{I}d(A))^c \neq \emptyset$. Let x be an arbitrary point of $(A \cup p\mathcal{I}d(A))^c$ so that $x \in A^c$ and $x \notin p\mathcal{I}d(A)$. Now $x \notin p\mathcal{I}d(A)$ implies that x is not a pre- \mathcal{I} -limit point of A and so there exists a pre- \mathcal{I} -open set N_x containing the point x and not containing any point of A other than x . Since $x \notin A$, N_x contains no point of A which implies that $N_x \subset (X - A)$. Again, N_x is a pre- \mathcal{I} -neighbourhood of each of its points. But N_x does not contain any point of A . Therefore, no point of N_x can be a pre- \mathcal{I} -limit point of A . Therefore, no point of N_x belongs to $p\mathcal{I}d(A)$. This implies that $N_x \subset (X - (A \cup p\mathcal{I}d(A)))$. Hence it follows that $x \in N_x \subset (X - (A \cup p\mathcal{I}d(A)))$ which implies that $X - (A \cup p\mathcal{I}d(A))$ is pre- \mathcal{I} -open. Hence $A \cup p\mathcal{I}d(A)$ is pre- \mathcal{I} -closed.

Theorem 3.8

Let (X, τ, \mathcal{I}) be an ideal space and $A \subset X$. Then $p\mathcal{I}cl(A) = A \cup p\mathcal{I}d(A)$.

Proof

By Theorem 3.7, $A \cup p\mathcal{I}d(A)$ is a pre- \mathcal{I} -closed set. Also as $A \subset A \cup p\mathcal{I}d(A)$, $A \cup p\mathcal{I}d(A)$ is a pre- \mathcal{I} -closed set containing A . Now $p\mathcal{I}cl(A)$ is the smallest pre- \mathcal{I} -closed set containing A . Therefore, $p\mathcal{I}cl(A) \subset A \cup p\mathcal{I}d(A)$.

Again $A \subset p\mathcal{I}cl(A)$ implies that $p\mathcal{I}d(A) \subset p\mathcal{I}d(p\mathcal{I}cl(A)) \subset p\mathcal{I}cl(A)$. Thus $A \subset p\mathcal{I}cl(A)$ and $p\mathcal{I}d(A) \subset p\mathcal{I}cl(A)$ which implies that $A \cup p\mathcal{I}d(A) \subset p\mathcal{I}cl(A)$. Hence $p\mathcal{I}cl(A) = A \cup p\mathcal{I}d(A)$.

It is obvious that $p\mathcal{I}cl(A) \subset cl(A)$. The converse is false as shown by the following Example 3.9.

Example 3.9

Consider the ideal space (X, τ, \mathcal{I}) where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. If $A = \{b\}$, then $cl(A) = X$ and $p\mathcal{I}cl(A) = \{b\}$. Hence $cl(A) \not\subset p\mathcal{I}cl(A)$.

Remark 3.10 If $p\mathcal{I}cl(A) = p\mathcal{I}cl(B)$, it does not imply that $A = B$. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{d\}, \{b, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{b\}, \{d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. Take $A = \{d\}, B = \{a, d\}$. Then $p\mathcal{I}cl(A) = p\mathcal{I}cl(B) = \{a, c, d\}$.

Theorem 3.11

Let A and B be subsets of an ideal topological space (X, τ, \mathcal{I}) . Then the following hold.

- (a) $p\mathcal{I}cl(A) \cup p\mathcal{I}cl(B) \subset p\mathcal{I}cl(A \cup B)$.
- (b) $p\mathcal{I}cl(A \cap B) \subset p\mathcal{I}cl(A) \cap p\mathcal{I}cl(B)$.

Proof

- (a) Now $A \subset (A \cup B)$ implies $p\mathcal{I}cl(A) \subset p\mathcal{I}cl(A \cup B)$. Also $B \subset (A \cup B)$ implies $p\mathcal{I}cl(B) \subset p\mathcal{I}cl(A \cup B)$. Thus, $p\mathcal{I}cl(A) \cup p\mathcal{I}cl(B) \subset p\mathcal{I}cl(A \cup B)$.
- (b) Since $A \cap B \subset A$ and $A \cap B \subset B$, $p\mathcal{I}cl(A \cap B) \subset p\mathcal{I}cl(A)$ and $p\mathcal{I}cl(A \cap B) \subset p\mathcal{I}cl(B)$. Hence $p\mathcal{I}cl(A \cap B) \subset p\mathcal{I}cl(A) \cap p\mathcal{I}cl(B)$.

4. PRE- \mathcal{I} -FRONTIER OF A SET

Definition 4.1

Let A be a subset of an ideal space (X, τ, \mathcal{I}) . $p\mathcal{I}cl(A) - p\mathcal{I}int(A)$ is said to be the pre- \mathcal{I} -frontier of $A \subset X$ and is denoted by $p\mathcal{I}F_r(A)$.

It is obvious that $p\mathcal{I}F_r(A) \subset F_r(A)$, the frontier of A . But the converse need not be true as shown by the following Example 4.2.

Example 4.2

Consider the ideal space (X, τ, \mathcal{I}) where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a, b\}, \{a, b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$. Let $A = \{a, b, d\}$. Then $int(A) = \{a, b\}$, $cl(A) = X$, $p\mathcal{I}cl(A) = X$, $p\mathcal{I}int(A) = \{a, b, d\}$, $p\mathcal{I}F_r(A) = \{c\}$ and $F_r(A) = \{c, d\}$. This shows that $F_r(A) \not\subset p\mathcal{I}F_r(A)$.

Theorem 4.3

Let A be a subset of an ideal topological space (X, τ, \mathcal{I}) . Then the following hold.

- (a) $p\mathcal{I}cl(A) = p\mathcal{I}int(A) \cup p\mathcal{I}F_r(A)$.
- (b) $p\mathcal{I}int(A) \cap p\mathcal{I}F_r(A) = \emptyset$.
- (c) $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) \cap p\mathcal{I}cl(X - A)$.

Proof

- (a) $p\mathcal{I}int(A) \cup p\mathcal{I}F_r(A) = p\mathcal{I}int(A) \cup (p\mathcal{I}cl(A) - p\mathcal{I}int(A)) = p\mathcal{I}cl(A)$, by Definition 4.1.
- (b) $p\mathcal{I}int(A) \cap p\mathcal{I}F_r(A) = p\mathcal{I}int(A) \cap (p\mathcal{I}cl(A) - p\mathcal{I}int(A)) = p\mathcal{I}int(A) \cap p\mathcal{I}cl(A) - (p\mathcal{I}int(A) \cap p\mathcal{I}int(A)) = p\mathcal{I}int(A) - p\mathcal{I}int(A) = \emptyset$.
- (c) $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}cl(A) \cap (X - p\mathcal{I}int(A)) = p\mathcal{I}cl(A) \cap p\mathcal{I}cl(X - A)$.

Definition 4.4

A subset A of a space (X, τ, \mathcal{I}) with an ideal \mathcal{I} is said to be pre- \mathcal{I} -regular if it is pre- \mathcal{I} -open and pre- \mathcal{I} -closed.

Theorem 4.5

Let (X, τ, \mathcal{I}) be an ideal topological space and $A \subset X$. Then $p\mathcal{I}F_r(A) = \emptyset$ if and only if A is pre- \mathcal{I} -regular.

Proof

The proof is clear.

Theorem 4.6

Let A be a subset of an ideal topological space (X, τ, \mathcal{I}) . Then the following hold.

- (a) $p\mathcal{I}F_r(A) = p\mathcal{I}f_r(X - A)$.
- (b) $A \in P\mathcal{I}O(X, \tau, \mathcal{I})$ if and only if $p\mathcal{I}F_r(A) \subset (X - A)$ (that is $A \cap p\mathcal{I}F_r(A) = \emptyset$.)
- (c) $A \in P\mathcal{I}C(X, \tau, \mathcal{I})$ if and only if $p\mathcal{I}F_r(A) \subset A$.

Proof

- (a) The proof is clear.
- (b) Let $A \in P\mathcal{I}O(X, \tau, \mathcal{I})$. Then by definition, $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}cl(A) - A$. Therefore, $A \cap p\mathcal{I}F_r(A) = A \cap (p\mathcal{I}cl(A) - A) = \emptyset$. Conversely, if $A \cap p\mathcal{I}F_r(A) = \emptyset$ implies that $(p\mathcal{I}cl(A) - p\mathcal{I}int(A)) \cap A = \emptyset$ which implies $(A \cap p\mathcal{I}cl(A)) \cap (X - p\mathcal{I}int(A)) = \emptyset$. Hence $A \cap (X - p\mathcal{I}int(A)) = \emptyset$ as $A \subset p\mathcal{I}cl(A)$. Thus, $A \subset X - (X - p\mathcal{I}int(A))$ implies that $A \subset p\mathcal{I}int(A)$. But on the otherhand, $p\mathcal{I}int(A) \subset A$. It follows that $A \in P\mathcal{I}O(X, \tau, \mathcal{I})$.
- (c) The proof follows from (a) and (b).

Remark 4.7 Let A and B be subsets of an ideal topological space (X, τ, \mathcal{I}) . Then $A \subset B$ does not imply that either $p\mathcal{I}F_r(A) \subset p\mathcal{I}F_r(B)$ (or) $p\mathcal{I}F_r(B) \subset p\mathcal{I}F_r(A)$. This can be verified by the following Example 4.8.

Example 4.8

Consider the ideal space (X, τ, \mathcal{I}) where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a, b\}, \{a, b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. Then $P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$.

Case (i)

Let $A = \{d\}$, $B = \{a, c, d\}$. Then $A \subset B$. Also, $p\mathcal{I}F_r(A) = \{d\}$ and $p\mathcal{I}F_r(B) = \{b\}$. Hence $p\mathcal{I}F_r(A) \not\subset p\mathcal{I}F_r(B)$.

Case (ii)

Let $A = \{d\}$, $B = \{c, d\}$ then $A \subset B$. Also, $p\mathcal{I}F_r(A) = \{d\}$ and $p\mathcal{I}F_r(B) = P\mathcal{I}cl(B) - p\mathcal{I}int(B) = \{c, d\} - \emptyset = \{c, d\}$. Hence $p\mathcal{I}F_r(B) \not\subset p\mathcal{I}F_r(A)$.

Theorem 4.9

For any subset A of an ideal topological space (X, τ, \mathcal{I}) , we have $p\mathcal{I}F_r(p\mathcal{I}F_r(A)) \subset p\mathcal{I}F_r(A)$.

Proof

By Theorem 4.3(c), $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) \cap p\mathcal{I}cl(X - A)$. Therefore $p\mathcal{I}F_r(p\mathcal{I}F_r(A)) = P\mathcal{I}cl(p\mathcal{I}F_r(A)) \cap p\mathcal{I}cl(X - p\mathcal{I}F_r(A)) \subset p\mathcal{I}cl(p\mathcal{I}F_r(A)) = p\mathcal{I}F_r(A)$, since $p\mathcal{I}F_r(A)$ is pre- \mathcal{I} -closed.

Theorem 4.10

Let A be a subset of an ideal topological space (X, τ, \mathcal{I}) . Then the following hold.

- (a) $p\mathcal{I}F_r(p\mathcal{I}int(A)) \subset p\mathcal{I}F_r(A)$.
- (b) $p\mathcal{I}F_r(P\mathcal{I}cl(A)) \subset p\mathcal{I}F_r(A)$.

Proof

- (a) $p\mathcal{I}F_r(p\mathcal{I}int(A)) = p\mathcal{I}cl(p\mathcal{I}int(A)) - p\mathcal{I}int(p\mathcal{I}int(A)) = p\mathcal{I}cl(p\mathcal{I}int(A)) - p\mathcal{I}int(A) \subset p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}F_r(A)$.

$$(b) p\mathcal{I}F_r(p\mathcal{I}cl(A)) = p\mathcal{I}cl(p\mathcal{I}cl(A)) - p\mathcal{I}int(p\mathcal{I}cl(A)) = p\mathcal{I}cl(A) - p\mathcal{I}int(p\mathcal{I}cl(A)) \subset p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}F_r(A).$$

Theorem 4.11

A subset A of an ideal topological space (X, τ, \mathcal{I}) is pre- \mathcal{I} -open if and only if $p\mathcal{I}F_r(A) = p\mathcal{I}d(A)$.

Proof

Let A be a pre- \mathcal{I} -open set then $p\mathcal{I}int(A) = A$. Now, $p\mathcal{I}F_r(A) = p\mathcal{I}cl(A) - p\mathcal{I}int(A) = p\mathcal{I}cl(A) - A$. Since $p\mathcal{I}cl(A) = A \cup p\mathcal{I}d(A)$, $p\mathcal{I}F_r(A) = (A \cup p\mathcal{I}d(A)) - A = p\mathcal{I}d(A)$.

Conversely, let $p\mathcal{I}F_r(A) = p\mathcal{I}d(A)$. Then $p\mathcal{I}d(A) = p\mathcal{I}cl(A) - p\mathcal{I}int(A) = (A \cup p\mathcal{I}d(A)) - p\mathcal{I}int(A)$. Hence $A - p\mathcal{I}int(A) = \emptyset$. Therefore, A is pre- \mathcal{I} -open.

Theorem 4.12

Let $\{A_\alpha | \alpha \in I\}$ be any family of subsets of an ideal topological space (X, τ, \mathcal{I}) . If $\cup_{\alpha \in I} p\mathcal{I}cl(A_\alpha)$ is pre- \mathcal{I} -closed, then $\cup_{\alpha \in I} p\mathcal{I}cl(A_\alpha) = p\mathcal{I}cl(\cup_{\alpha \in I} A_\alpha)$.

Proof

Since $A_\alpha \subset \cup_{\alpha \in I} A_\alpha$, $p\mathcal{I}cl(A_\alpha) \subset p\mathcal{I}cl(\cup_{\alpha \in I} A_\alpha)$ and hence $\cup_{\alpha \in I} p\mathcal{I}cl(A_\alpha) \subset p\mathcal{I}cl(\cup_{\alpha \in I} A_\alpha)$. We will show that $p\mathcal{I}cl(\cup_{\alpha \in I} A_\alpha) \subset \cup_{\alpha \in I} p\mathcal{I}cl(A_\alpha)$. Let $x \in p\mathcal{I}cl(\cup_{\alpha \in I} A_\alpha)$. Since $\cup_{\alpha \in I} p\mathcal{I}cl(A_\alpha)$ is pre- \mathcal{I} -closed it contains all its pre- \mathcal{I} -limit points of $\cup_{\alpha \in I} p\mathcal{I}cl(A_\alpha)$ and so, there exists a pre- \mathcal{I} -neighbourhood U_x of x such that $U_x \cap (\cup_{\alpha \in I} p\mathcal{I}cl(A_\alpha)) = \emptyset$. This implies that $U_x \cap p\mathcal{I}cl(A_\alpha) = \emptyset$ for every $\alpha \in I$ and hence $U_x \cap A_\alpha = \emptyset$ which is a contradiction to $x \in p\mathcal{I}cl(\cup_{\alpha \in I} A_\alpha)$. Therefore, $p\mathcal{I}cl(\cup_{\alpha \in I} A_\alpha) \subset \cup_{\alpha \in I} p\mathcal{I}cl(A_\alpha)$ which completes the proof.

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