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RESEARCH ARTICLE

Sequence of Functions Involving the Product of S-Functions

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ABSTRACT

A remarkably large number of operational techniques have drawn the attention of several researchers in the study of sequence of functions and polynomials. In this sequel, here, we aim to introduce a new sequence of functions involving the S-functions $S_{p,q}^{\varepsilon,\zeta,\sigma,\tau,\rho}(a_1,...,a_p;b_1,...b_q;x)$ by using operational techniques. Some generating relations and finite summation formula of the sequence presented here are also considered.
Keywords: Polynomials, Functions, S-functions, Finite summation formula, Operational techniques.

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1. INTRODUCTION

Recently, a remarkable interest has been developed in the study of operational techniques due to their importance in many fields of engineering and mathematical physics. The sequences of functions play an important role in approximation theory. They can be used to show a solution if a differential equation exists. Therefore, a large body of research in the development of these sequences has been published in the literature.

In the literature, there are numerous sequences of functions which are widely used in physics, mathematics as well as in engineering. Sequences of functions are also used to solve some differential equations in a rather efficient way. Here, we introduce and investigate further computable extensions of the sequence of functions involving the $S_{p,q}^{\varepsilon,\zeta,\sigma,\tau,\rho}(a_1,...,a_p;b_1,...b_q;x)$ by using operational techniques. The generating relations and finite summation formulas in terms of S-functions, are written in a compact and easily computable form as in section 2 and 3. Finally, some special cases and concluding remarks are discussed in section 4 and 5.

To investigate the sequence of functions, we recall some facts about the S-functions and its generalizations. Let us begin with few notions and facts related to the S-functions. In this presentation, we follow mainly the review articles. The analysis is shown in equation (1) to (36)

Recently [7] introduced and studied some fundamental properties and characteristics of the S-function in their paper defined as,

$$S_{(p,q)}^{\alpha,\beta,\gamma,\tau,k}(a_1,...,a_p;b_1,..,b_q;x)=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n(\gamma)_{n\tau,k}}{(b_1)_n...(b_p)_n\Gamma_k(n\alpha+\beta)}\frac{x^n}{n!'} \tag{1}$$

$k\in R;\alpha,\beta,\gamma,\tau\in C;\Re(\alpha)>0,a_i(i=1,2,...,p),b_j(j=1,2,...,q),\Re(\alpha)>k\Re(\tau)$ and $p<q+1.(\lambda)_n$ denote the Pochhammer symbol defined (for $\lambda\in C$) in terms of $\Gamma(z)$ in the following way:

$$(\lambda)_n:=\begin{cases}1 & \text{(n=0)}\\ \lambda(\lambda+1)...(\lambda+n-1) & \text{(n}\in N:\{1,2,3,...\})\end{cases}$$
$$=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\ \ (\lambda\in C/Z_0^-) \tag{2}$$

and Z_0^- denotes the set of non-positive integers.

For our purpose, we begin by recalling some known functions and earlier works. In 1971, [1, 2] gave the Rodrigues formula for the generalized Lagurre polynomials defined as:

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$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}exp(p_k(x))D^n[x^{\alpha+n}exp(-p_k(x))] \tag{3}$$

where $p_k(x)$ is a polynomial in x of degree k. [3, 4] also proved the following relation for equation (3) defined as:

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!}x^{-\alpha-n}exp(p_k(x))T_s^n[x^{\alpha}exp(-p_k(x))] \tag{4}$$

where s is constant and $T_s \equiv x(s+xD)$.

[5, 6] provides a sequence of functions $V_n^{(\alpha)}(x;a,k,s)$ defined as:

$$V_n^{(\alpha)}(x;a,k,s) = \frac{x^{-\alpha}}{n!}exp\{p_k(x)\}\theta^n[x^{\alpha}exp\{-p_k(x)\}] \tag{5}$$

By employing the operator $\theta \equiv x^a(s+xD)$, where s is constant and $p_k(x)$is a polynomial in x of degree k, a new sequence of function $\{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,...,k_r,s)\}_{n=0}^{\infty}$ is introduced in this paper as:

$$\begin{aligned} &\{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,...,k_r,s) \\ &= \frac{1}{n!}x^{-\alpha}\prod_{j=1}^{r}S_{(p_j;q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};P_{k_j}(x)\right] \\ &(T_x^{a,s})^n\left(x^{\alpha}\prod_{j=1}^{r}S_{(p_j;q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}(x)\right]\right) \end{aligned} \tag{6}$$

where $T_x^{a,s} \equiv x^a(s+xD)$, $D \equiv \frac{d}{dx}$, a and s are constants, k is finite and non-negative integer, $p_k(x)$ is a polynomial in x of degree k and $S_{(p,q)}^{\varepsilon,\zeta,\sigma,\tau,\rho}(a_1,...,a_p;b_1,...b_q;x)$ is a S-function, which is defined in equation (1).

Some generating relations and finite summation formula of class of polynomials or sequence of function have been obtained by using the properties of the differential operators. $T_x^{a,s} \equiv x^a(s+xD)$, where $D \equiv \frac{d}{dx}$, is based on the work of [4-6]. Some useful operational techniques are given below:

$$exp(tT_x^{a,s})\left(x^{\beta}f(x)\right) = x^{\beta}(1-ax^at)^{-(\frac{\beta+s}{a})}f\left(x(1-ax^at)^{-1/a}\right), \tag{7}$$

$$exp(tT_x^{a,s})\left(x^{\alpha-an}f(x)\right) = x^{\alpha}(1+at)^{-1+(\frac{\alpha+s}{a})}f\left(x(1+at)^{1/a}\right), \tag{8}$$

$$(T_x^{a,s})^n(xuv) = x\sum_{m=0}^{\infty}\binom{n}{m}(T_x^{a,s})^{n-m}(v)(T_x^{a,1})^m(u), \tag{9}$$

$$\begin{aligned} &(1+xD)(1+a+xD)(1+2a+xD)(1+3a+xD)...(1+(m-1)a+xD)x^{\beta-1} \\ &= a^m\left(\frac{\beta}{a}\right)_m x^{\beta-1}, \end{aligned} \tag{10}$$

$$(1-at)^{\frac{-\alpha}{a}} = (1-at)^{\frac{-\beta}{a}}\sum_{m=0}^{\infty}\left(\frac{\alpha-\beta}{a}\right)_m\frac{(at)^m}{m!}. \tag{11}$$

2. GENERATING RELATIONS

First generating relation:

$$\sum_{n=0}^{\infty} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,...,k_r,s)x^{-an}t^n$$
$$= (1-at)^{-(\frac{\alpha+s}{a})}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j,p_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};P_{k_j}(x)\right] \tag{12}$$
$$K_1,...k_r \times \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}\left(x(1-at)^{-1/a}\right)\right].$$

Second generating relation

$$\sum_{n=0}^{\infty} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha-an)}(x;a,k_1,...,k_r,s)x^{-an}t^n$$
$$= (1+at)^{-1+(\frac{\alpha+s}{a})}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};P_{k_j}(x)\right] \tag{13}$$
$$\times \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}\left(x(1+at)^{1/a}\right)\right].$$

Third generating relation

$$\sum_{m=0}^{\infty}\binom{m+n}{n} V_{m+n}^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,...,k_r,s)x^{-am}t^m$$
$$= (1-at)^{-(\frac{\alpha+s}{a})}$$
$$\times \frac{\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};P_{k_j}(x)\right]}{\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};P_{k_j}\left(x(1-at)^{-1/a}\right)\right]} \tag{14}$$
$$\times V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x(1-at)^{-1/a};a,k_1,...,k_r,s)$$

Proof of first generating relation:

From equation (6), we obtain:

$$\sum_{n=0}^{\infty} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,...,k_r,s)t^n$$
$$= x^{-\alpha}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};P_{k_j}(x)\right] \tag{15}$$
$$\times exp(tT_x^{a,s})\left(x^{\alpha}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}(x)\right]\right)$$

Using operational technique of equation (7), above equation (15) reduces to:

$$\sum_{n=0}^{\infty} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,...,k_r,s)t^n$$
$$= (1-ax^a t)^{-(\frac{\alpha+s}{a})}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};P_{k_j}(x)\right] \tag{16}$$
$$\times \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}\left(x(1-ax^a t)^{-(1/a)}\right)\right]$$

Replacing t by tx^{-a}, equation (12) is obtained.

Proof of second generating relation:

Again from equation (6), we have:

$$\begin{aligned}&\sum_{n=0}^{\infty} x^{-an} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha-an)}(x;a,k_1,\dots,k_r,s)t^n \\ &= x^{-\alpha} \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};P_{k_j}(x)\right] \\ &\times exp(tT_x^{a,s})\left(x^{\alpha-an}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};-P_{k_j}(x)\right]\right)\end{aligned} \tag{17}$$

Applying the operational technique of equation (8), we get:

$$\begin{aligned}&\sum_{n=0}^{\infty} x^{-an} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha-an)}(x;a,k_1,\dots,k_r,s)t^n \\ &= (1+at)^{\frac{\alpha+s}{a}-1} \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};P_{k_j}(x)\right] \\ &\times \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};-P_{k_j}\left(x(1+at)^{1/a}\right)\right]\end{aligned} \tag{18}$$

which is desired.

Proof of third generating relation:

We can write equation (6) as:

$$\begin{aligned}&(T_x^{a,s})^n\left[x^a \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};-P_{k_j}(x)\right]\right] \\ &= n!x^{\alpha}\frac{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,\dots,k_r,s)}{\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};P_{k_j}(x)\right]}\end{aligned} \tag{19}$$

$$\begin{aligned}&expt(t(T_x^{a,s}))\left((T_x^{a,s})^n\left[x^{\alpha} \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};-P_{k_j}(x)\right]\right]\right) \\ &= n!expt(tT_x^{a,s})\left[x^{\alpha}\frac{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,\dots,k_r,s)}{\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};P_{k_j}(x)\right]}\right]\end{aligned} \tag{20}$$

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{t^m}{m!}(T_x^{a,s})^{m+n}\left(x^a \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};-P_{k_j}(x)\right]\right) \\ &= n!expt(tT_x^{a,s})\left[x^{\alpha}\frac{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,\dots,k_r,s)}{\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j};b_{1_j},\dots,b_{q_j};P_{k_j}(x)\right]}\right]\end{aligned} \tag{21}$$

Using the operational technique of equation (7), above equation can be written as:

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{t^m}{m!}(T_x^{a,s})^{m+n}\left(x^a \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j,p_j}\left[a_{1\,j},\dots,a_{p\,j};b_{1\,j},\dots,b_{q\,j};-P_{k\,j}(x)\right]\right) \\ &= n!x^{\alpha}(1-ax^{a}t)^{-\left(\frac{\alpha+s}{a}\right)} \\ &\times\frac{V_n^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha}\left(x(1-ax^{a}t)^{-1/a};a,k_1,\dots,k_{[}r],s\right)}{\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j,p_j}\left[a_{1\,j},\dots,a_{p\,j};b_{1\,j},\dots,b_{q\,j};-P_{k\,j}\left(x(1-ax^{a}t)^{-1/a}\right)\right]}\end{aligned} \tag{22}$$

Using equation (30), above equation gives:

$$\begin{aligned}\sum_{m=0} \frac{t^m (m+n)!}{m!n!} x^a \frac{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x)}{\prod_{j=1}^r S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j,p_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}(x)\right]} \\ = x^{\alpha}(1-ax^a t)^{-(\alpha+\frac{s}{a})} \\ \times \frac{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}\left(x(1-ax^a t)^{-1/a};a,k_1,...,k_[r],s\right)}{\prod_{j=1}^r S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j,p_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}\left(x(1-ax^a t)^{-1/a}\right)\right]}\end{aligned} \tag{23}$$

Therefore

$$\begin{aligned}\sum_{m=0} V_{m+n}^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}(x;a,k_1,...,k_r,s)t^m \\ = (1-ax^a t)^{-(\alpha+\frac{s}{a})} \\ \times \frac{\prod_{j=1}^r S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j,p_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}(x)\right]}{\prod_{j=1}^r S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j,p_j}\left[a_{1_j},...,a_{p_j};b_{1_j},...,b_{q_j};-P_{k_j}\left(x(1-ax^a t)^{-1/a}\right)\right]} \\ \times V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j;q_j;\alpha)}\left(x(1-ax^a t)^{-1/a};a,k_1,...,k_[r],s\right)\end{aligned} \tag{24}$$

Replacing t by tx^{-a}, gives the result in equation (14).

Remark 1 If we give some suitable parametric replacement in equations (12), (13) and (14) respectively, then we can see the known results ([1]).

3. FINITE SUMMATION FORMULAS

3.1. First finite summation formula

$$\begin{aligned}V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\alpha)}(x;a,k,s) \\ = \sum_{m=0}^{n} \frac{1}{m!}(ax^a)^m \left(\frac{\alpha}{a}\right)_m V_{n-m}^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;0)}(x;a,k,s)\end{aligned} \tag{25}$$

3.2. Second finite summation formula

$$\begin{aligned}V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\alpha)}(x;a,k_1,...,k_r,s) \\ = \sum_{m=0}^{n} \frac{1}{m!}(ax^a)^m \left(\alpha-\frac{\beta}{a}\right)_m V_{n-m}^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\beta)}(x;a,k_1,...,k_r,s)\end{aligned} \tag{26}$$

Proof of first finite summation formula:

From equation (6), we have:

$$\begin{aligned}V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\alpha)}(x;a,k_1,...,k_r,s) \\ = \frac{1}{n!}x^{-\alpha}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};p_{k_j}(x)\right] \\ \times (T_x^{a,s})^n \left(xx^{\alpha-1}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};-p_{k_j}(x)\right]\right)\end{aligned} \tag{27}$$

Using the operational technique in equation (9), we have:

$$\begin{aligned} & V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\alpha)}(x;a,k_1,\dots,k_r,s) \\ &= \frac{1}{n!}x^{-\alpha}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};p_{k_j}(x)\right]\times\sum_{m=0}^{\infty}(nm) \\ &\times (T_x^{a,s})^{n-m}\left(\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};-p_{k_j}(x)\right]\right)(T_x^{a,1})^m(x^{\alpha-1}) \\ &= \frac{1}{n!}x^{-\alpha}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};p_{k_j}(x)\right]x\sum_{m=0}^{\infty}\frac{n!}{m!(n-m)!}x^{a(n-m)} \\ &\times\left[(s+xD)(s+a+xD)(s+2a+xD)\dots(s+(n-m-1)a+xD)\right] \\ &\times\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};-p_{k_j}(x)\right]x^{am} \\ &\times\left[(1+xD)(1+a+xD)(1+2a+xD)\dots(1+(m-1)a+xD)\right](x^{\alpha-1}) \end{aligned} \tag{28}$$

Using the result in (10), we have:

$$\begin{aligned} & V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\alpha)}(x;a,k_1,\dots,k_r,s) \\ &= \prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};p_{k_j}(x)\right] \\ &\times\sum_{m=0}^{\infty}\frac{1}{m!(n-m)!}x^{an}\prod_{i=0}^{n-m-1}(s+ia+xD) \\ &\times\left(\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};-p_{k_j}(x)\right]\right)a^m\left(\frac{\alpha}{a}\right)_m \end{aligned} \tag{29}$$

Putting $\alpha = 0$ and replacing n by $n-m$ in equation (27), we get:

$$\begin{aligned} & V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;0)}(x;a,k_1,\dots,k_r,s) \\ &= \frac{1}{(n-m)!}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};p_{k_j}(x)\right] \\ &\times(T_x^{a,s})^{n-m}\left(\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};-p_{k_j}(x)\right]\right) \end{aligned} \tag{30}$$

$$\begin{aligned} &\Longrightarrow \frac{1}{(n-m)!}(T_x^{a,s})^{n-m}\left(\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};-p_{k_j}(x)\right]\right) \\ &= \frac{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;0)}(x;a,k_1,\dots,k_r,s)}{\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};p_{k_j}(x)\right]} \end{aligned} \tag{31}$$

This gives

$$\begin{aligned} & \frac{1}{(n-m)!}\prod_{i=0}^{n-m-1}(s+ia+xD) \\ &\times\left(\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};-p_{k_j}(x)\right]\right) \\ &= x^{a(m-n)}\frac{V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;0)}(x;a,k_1,\dots,k_r,s)}{\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},\dots,a_{p_j},\dots,b_{1_j},\dots,b_{q_j};p_{k_j}(x)\right]} \end{aligned} \tag{32}$$

From equation (29) and (32) we have the main result.

Proof of second finite summation formula:

Equation (6)can be written as:

$$\sum_{n=0}^{\infty} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\alpha)}(x;a,k_1,...,k_r,s)t^n$$
$$= x^{-a}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};p_{k_j}(x)\right] \tag{33}$$
$$exp\left[tT_x^{(a,s)}\right]\left(x^{alpha}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};-p_{k_j}(x)\right]\right)$$

Applying the result from equation (7), equation (33) gets reduced to:

$$\sum_{n=0}^{\infty} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\alpha)}(x;a,k_1,...,k_r,s)t^n$$
$$= (1-ax^a t)^{-(\alpha+\frac{s}{a})}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};p_{k_j}(x)\right] \tag{34}$$
$$\times\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};-p_{k_j}\left(x((1-ax^a t)^{-1/a})\right)\right]$$

Applying equation (11); equation (34) reduces to:

$$\sum_{n=0}^{\infty} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;\alpha)}(x;a,k_1,...,k_r,s)t^n$$
$$= (1-ax^a t)^{-(\beta+\frac{s}{a})}\sum_{m=0}^{\infty}\left(\alpha-\frac{\beta}{a}\right)_m \frac{(ax^a t)^m}{m!}$$
$$\times\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};p_{k_j}(x)\right]$$
$$\times\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};-p_{k_j}\left(x((1-ax^a t)^{-1/a})\right)\right]$$
$$= \sum_{m=0}^{\infty}\left(\alpha-\frac{\beta}{a}\right)_m \frac{(ax^a t)^m}{m!}x^{-\beta}$$
$$\times\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};p_{k_j}(x)\right]$$
$$\times exp(tT_x^{a,s})\left(x^{\beta}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};-p_{k_j}(x)\right]\right)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\alpha-\frac{\beta}{a}\right)_m \frac{(ax^a)^m t^{n+m}}{m!}x^{-\beta}$$
$$\times\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};p_{k_j}(x)\right]$$
$$\times(T_x^{a,s})^n\left(x^{\beta}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};-p_{k_j}(x)\right]\right)$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\alpha-\frac{\beta}{a}\right)_m \frac{(ax^a)^m t^n}{m!}x^{-\beta}$$
$$\times\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};p_{k_j}(x)\right] \tag{35}$$
$$\times(T_x^{a,s})^{n-m}\left(x^{\beta}\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};-p_{k_j}(x)\right]\right)$$

Now equating the coefficient of t^n , we get:

$$\begin{aligned} V_n^{(\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j;p_j,q_j;0)}&(x;a,k_1,...,k_r,s) \\ &= \sum_{m=0}^{\infty}\left(\alpha-\frac{\beta}{a}\right)_m \frac{(ax^a)^m}{m!(n-m)!}x^{-\beta} \\ \times\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j} &\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};p_{k_j}(x)\right] \\ \times(T_x^{a,s})^{n-m}\left(x^\beta\prod_{j=1}^{r} S_{(p_j,q_j)}^{\varepsilon_j,\zeta_j,\sigma_j,\tau_j,\rho_j}\right.&\left.\left[a_{1_j},...,a_{p_j},...,b_{1_j},...,b_{q_j};-p_{k_j}(x)\right]\right) \end{aligned} \tag{36}$$

Using equation (6) in (36), we have the result as shown in (26).

4. SPECIAL CASES

(1) When $p=q=0$ the S-functions get reduced to generalized k-Mittag-Leffler function, defined by [8]

$$S_{0,0}^{\alpha,\beta,\gamma,\tau,k}(-;-;x)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)n!}=E_{k,\alpha,\beta}^{\gamma,\tau}(x)$$

where $R(\alpha/k-\tau)>0$, all the results in section 2 and 3 are reduced to the new results involving $E_{k,\alpha,\beta}^{\gamma,\tau}(x)$

(2) If $\tau=q=1$ the S-functions get reduced to generalized K-function, defined by [9]

$$\begin{aligned} S_{p,q}^{\alpha,\beta,\gamma,1,1}(a_1,...,a_p;b_1,...,b_q;x) &= \sum_{n=0}^{\infty}\frac{(a_1)_n,...,(a_p)_n(\gamma)_n}{(b_1)_n,...,(b_q)_n\Gamma(n\alpha+\beta)}\frac{x^n}{n!} \\ &= K_{p,q}^{\alpha,\beta}(a_1,...,a_p;b_1,...,b_q;x) \end{aligned}$$

where $R(\alpha)>p-q$, then all the above results in section 2 and 3 are reduced to the new one involving K-function $K_{p,q}^{\alpha,\beta}(a_1,...,a_p;b_1,...,b_q;x)$.

(3) When $\tau=q=\gamma=1$ the S-functions are reduced to generalized M-Series, defined by [10]

$$\begin{aligned} S_{p,q}^{\alpha,\beta,1,1,1}(a_1,...,a_p;b_1,...,b_q;x) &= \sum_{n=0}^{\infty}\frac{(a_1)_n,...,(a_p)_n}{(b_1)_n,...,(b_q)_n\Gamma(n\alpha+\beta)}\frac{x^n}{n!} \\ &= M_{p,q}^{\alpha,\beta,\gamma}(a_1,...,a_p;b_1,...,b_q;x) \end{aligned}$$

where $R(\alpha)>p-q-1$, then all the results in section 2 and 3 are reduced to the new one involving M-Series $M_{p,q}^{\alpha,\beta,\gamma}(a_1,...,a_p;b_1,...,b_q;x)$

(4) If we choose $\gamma=\lambda=1;m=1$ then all the results in equations (12), (13), (14), (25) and (26) reduced to the work of [1, 10, 11, 12].

(5) If we choose $\gamma=\lambda=\alpha_j=\beta_j=1;m=1$, the generalized Multi-Index Mittag- Leffler function reduced to $exp(z)$ i.e, $E_{1,1}^{1,1}exp(z)$, then all the results in equation (12), (13), (14), (25) and (26) are reduced to the new one involving $exp(x)$.

(6) Again if we choose $\gamma=\lambda=\alpha_j=\beta_j=1;\alpha_j=2;m=1$, the generalized Multi-Index Mittag-Leffler function reduced to $cosh(z)$ i.e,$E_{2,1}^{1,1}=cosh(z)$, then all the results in equation (12), (13), (14), (25) and (26) reduced to the new one involving $cosh(x)$.

5. CONCLUSION

In this paper, we have presented a new sequence of functions involving the S-function by using operational techniques. With the help of our main sequence formula, some generating relations and definite summation formula of the sequence are also presented here. Our sequence formula is important due to presence of $S_{p,q}^{\varepsilon,\zeta,\sigma,\tau,\rho}(a_1,...,a_p;b_1,...b_q;x)$. On account of the most general nature of the $S_{p,q}^{\varepsilon,\zeta,\sigma,\tau,\rho}(a_1,...,a_p;b_1,...b_q;x)$ a large number of sequences and polynomials involving simpler functions can be easily obtained as their special cases but due to lack of space we cannot mention here.

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