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RESEARCH ARTICLE

Solving Anisotropic Model Equations for Media Layer of Diseased Arterial Tissues

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ABSTRACT

During the last decade scientists have concentrated their efforts on modeling the arterial tissues behaviour. On the basis of nonlinear mechanics they developed constitutive models of arterial vessels. Media layer is the main layer of the arterial tissues bearing biomechanical load of the diseased arteries. Holzapfel anisotropic model for media layer of arterial tissues is considered in this paper. The strain-energy function consists of the strain-energy function of the ground matrix (elastin) and the strain-energy function of the embedded two collagen fibers families. The system of the obtained equations for Cauchy stress components is solved for definite constants and the adequacy of the applied model and the assumed constants is proved here.

Keywords: Arterial tissues, Biomechanics, Anisotropic model, Cauchy stress, Tensor components.

1. INTRODUCTION

During the last decade mechanobiology of soft biological tissues has achieved the focus of attention. Scientists have concentrated their efforts on modelling the arterial tissues behaviour. They developed their studies on the basis of non-linear elasticity laws. The elastic properties of the arterial tissues have been modelled considering invariants associated with the directions of the collagen fibers and their dispersion. In this way structural aspect of the tissues are considered.

At low strains such as physiological pressures in arterial vessels, it is the media layer that significantly contributes the wall stiffness.

One of the attempts in the field of constitutative modeling of diseased arterial tissues comprises the reporting of passive reaction of the elastin by means of strain energy function of the ground matrix and the active reaction –reinforcement, by means of strain energy function of collagen fibers' families.

Holzapfel has used this approach and has developed an anisotropic model reporting the mechanobiological behavior of media layer of diseased arterial tissues [2,3,4,6]. The strain-energy function consists of the strain-energy function of the ground matrix and the strain-energy function of the embedded two collagen fiber families.

For modelling the softening behaviour of the tissues, an established hyperelastic constitutive model was extended with damage function [12]. A hyperelastic strain-energy function was multiplied by a reduction factor that governs the softening behavior of the arteries.

[5] proposed a damage model for arteries using fourth-order weigthing tensor linked to the Ogden material model. The model is complicated and does not account for anisotropy.

[1] propose a damage model which is applied under conditions of surgery interventions. The model accounts for the damage in three structural components of of the aortic wall, including the elastin matrix, the collagen fibers and smooth muscle cells. This needs an estimation of total 21 parameters.

Holazapfel anisotropic model equations valid for intimal and adventitial layers of biaxial stretched cerebral tissues have been solved analytically by [10,14]. Holzapfel anisotropic model for adventita cerebral layer has been solved and circumferential and axial stretches have been calculated by [11]. Appropriate values are obtained for the stretch ratios in two directions that proves the adequacy of applied models and assumed constants.

The Holzapfel anisotropic model for media layer of diseased arterial tissues is considered in this paper. The system of the obtained equations for Cauchy stress components is solved analytically for definite constants. The

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stretch ratios in three directions are obtained in an appropriate range that proves the adequacy of the applied model and the constants taken into account.

2. SOLVING SYSTEM OF EQUATIONS FOR CAUCHY STRESS COMPONENTS

[6] proposed an equation for strain-energy function reporting biomechanical properties of arterial tissues. Adventita is about 10 percent for large elastic arteries and the intima of healthy arteries is not of mechanical interest. Media is the mechanically relevant layer within physiological loading domain [4,13].

The strain-energy function of arterial tissues consists of the strain-energy function of the non-collagenous ground substance (elastin) $\bar{\psi}_g$ and strain-energy of embedded two collagen fibers families $\bar{\psi}_f$.

$$\bar{\psi} = \bar{\psi}_g + \bar{\psi}_f \tag{1}$$

$$\bar{\psi} = \frac{\mu}{2}(\bar{I}_1 - 3) + \Sigma_{i=4,6}\frac{k_1}{2k_2}\left[exp\left[k_2(\bar{I}_i - 1)^2\right] - 1\right] \tag{2}$$

$$\bar{I}_1 = tr\bar{\mathbf{C}}, \quad \bar{I}_4 = \hat{\mathbf{a}}_0.\bar{\mathbf{C}}.\hat{\mathbf{a}}_0, \quad \bar{I}_6 = \hat{\mathbf{b}}_0.\bar{\mathbf{C}}.\hat{\mathbf{b}}_0 \tag{3}$$

where $\bar{I}_1$ is the first modified isotropic invariant and $\bar{I}_4$ and $\bar{I}_6$ are modified invariants representing the square of stretches along the two families of collagen fibers, $\bar{\mathbf{C}}$ is the modified right Cauchy-Green tensor, $\hat{\mathbf{a}}_0$ and $\hat{\mathbf{b}}_0$ are the unit vectors in the directions of the two families of fibers in the reference configuration, μ, k_1 k_1 are material parameters.

The Cauchy stress is presented by the following equation:

$$\begin{aligned}\sigma = \mu dev\bar{B} + 2\bar{\psi}_4 dev(\mathbf{a}\bigotimes\mathbf{a})\\ +2\bar{\psi}_6 dev(\mathbf{b}\bigotimes\mathbf{b})\end{aligned} \tag{4}$$

where $\bar{\psi}_i = \frac{\partial\bar{\psi}}{\partial\bar{I}_i}$, i=4, 6, $\hat{a}$ and $\hat{b}$ are the unit vectors in the directions of the two families of fibers in the deformed configuration expressed by equation (5):

$$\mathbf{a} = \mathbf{F}\hat{\mathbf{a}}_\mathbf{0}, \quad \mathbf{b} = \mathbf{F}\hat{\mathbf{b}}_\mathbf{0} \tag{5}$$

Deviatoric part is equalled to the following equation [7]:

$$dev(\bullet) = (\bullet) - \frac{1}{3}\left[(\bullet):\mathbf{I}\right]\mathbf{I} \tag{6}$$

where $\mathbf{I}$ is identity matrix.

The derivatives of strain-energy function in reference of $\bar{I}_4$ and $\bar{I}_6$ will be the following:

$$\frac{\partial\bar{\psi}}{\partial\bar{I}_4} = k_1(\bar{I}_4 - 1)exp\left[k_2(\bar{I}_4 - 1)^2\right] \tag{7}$$

$$\frac{\partial\bar{\psi}}{\partial\bar{I}_6} = k_1(\bar{I}_6 - 1)exp\left[k_2(\bar{I}_6 - 1)^2\right] \tag{8}$$

After substituting equation (5-8) in equation (4), the following equation is obtained for Cauchy stress σ :

$$\begin{aligned}\bar{\sigma} = \mu\left[\bar{\mathbf{F}}\bar{\mathbf{F}}^T - \frac{1}{3}\left[\bar{\mathbf{F}}\bar{\mathbf{F}}^T:\mathbf{I}\right]\mathbf{I}\right] + 2k_1(\bar{I}_4 - 1)exp\left[k_2(\bar{I}_4 - 1)^2\right]\\ \left[(\mathbf{Fa_0}\bigotimes\mathbf{Fa_0}) - \frac{\mathbf{1}}{\mathbf{3}}\left[(\mathbf{Fa_0}\bigotimes\mathbf{Fa_0}):\mathbf{I}\right]\mathbf{I}\right]\\ +2k_1(\bar{I}_6 - 1)exp\left[k_2(\bar{I}_6 - 1)^2\right]\\ \left[(\mathbf{Fb_0}\bigotimes\mathbf{Fb_0}) - \frac{\mathbf{1}}{\mathbf{3}}\left[(\mathbf{Fb_0}\bigotimes\mathbf{Fb_0}):\mathbf{I}\right]\mathbf{I}\right]\end{aligned} \tag{9}$$

where $\bar{\mathbf{F}}$ is the modified deformation gradient, $\bar{\mathbf{F}} = J^{-\frac{1}{3}}\mathbf{F}$, $J = det\ \mathbf{F} = \lambda_1\lambda_2\lambda_3$ and $\lambda_1,\ \ \lambda_2,\ \ \lambda_3$ are the stretches in the three directions The modified invariants $\bar{I_4}$ and $\bar{I_6}$ can be expressed as follows:

$$\bar{I_4} = \hat{\mathbf{a}}_0.\bar{\mathbf{F}}^T\bar{\mathbf{F}}.\hat{\mathbf{a}}_0 = \hat{\mathbf{a}}_0.\left(J^{-\frac{2}{3}}\mathbf{F}^{\mathbf{T}}\mathbf{F}\right)\hat{\mathbf{a}}_0 \tag{10}$$

$$\bar{I_6} = \hat{\mathbf{b}}_0.\bar{\mathbf{F}}^T\bar{\mathbf{F}}.\hat{\mathbf{b}}_0 = \hat{\mathbf{b}}_0.\left(J^{-\frac{2}{3}}\mathbf{F}^{\mathbf{T}}\mathbf{F}\right).\hat{\mathbf{b}}_0 \tag{11}$$

The Cauchy stress is calculated as a sum of $\mathbf{W_1},\ \ \mathbf{W_2},\ \ \mathbf{W_3}$.

$$\sigma = \mathbf{W_1} + \mathbf{W_2} + \mathbf{W_3} \tag{12}$$

$$\mathbf{W_1} = \begin{bmatrix} \frac{2\mu}{3}\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} & 0 & 0 \\ 0 & \frac{2\mu}{3}\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} & 0 \\ 0 & 0 & \frac{2\mu}{3}\lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} \end{bmatrix} \tag{13}$$

$$\begin{aligned} \mathbf{W_2} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} .\mathbf{exp} \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{12} & 0 \\ 0 & 0 & \alpha_{13} \end{bmatrix} \\ .\begin{bmatrix} \frac{2}{3}\hat{\mathbf{a}}_{0_1}^2\lambda_1^2 & 0 & 0 \\ 0 & \frac{2}{3}\hat{\mathbf{a}}_{0_2}^2\lambda_2^2 & 0 \\ 0 & 0 & \frac{2}{3}\hat{\mathbf{a}}_{0_3}^2\lambda_3^2 \end{bmatrix} \end{aligned} \tag{14}$$

where, $\alpha_1 = 2k_1\left(\hat{\mathbf{a}}_{0_1}^2\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} - 1\right)$,
$\alpha_2 = 2k_1\left(\hat{\mathbf{a}}_{0_2}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} - 1\right)$,
$\alpha_3 = 2k_1\left(\hat{\mathbf{a}}_{0_3}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} - 1\right)$,
$\alpha_{11} = k_2\left(\hat{\mathbf{a}}_{0_1}^2\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} - 1\right)^2$,
$\alpha_{12} = k_2\left(\hat{\mathbf{a}}_{0_2}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} - 1\right)^2$,
$\alpha_{13} = k_2\left(\hat{\mathbf{a}}_{0_3}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} - 1\right)^2$

$$\begin{aligned} \mathbf{W_2} = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix} .\mathbf{exp} \begin{bmatrix} \beta_{11} & 0 & 0 \\ 0 & \beta_{12} & 0 \\ 0 & 0 & \beta_{13} \end{bmatrix} \\ .\begin{bmatrix} \frac{2}{3}\hat{\mathbf{b}}_{0_1}^2\lambda_1^2 & 0 & 0 \\ 0 & \frac{2}{3}\hat{\mathbf{b}}_{0_2}^2\lambda_2^2 & 0 \\ 0 & 0 & \frac{2}{3}\hat{\mathbf{b}}_{0_3}^2\lambda_3^2 \end{bmatrix} \end{aligned} \tag{15}$$

where $\beta_1 = 2k_1\left(\hat{\mathbf{b}}_{0_1}^2\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} - 1\right)$,
$\beta_2 = 2k_1\left(\hat{\mathbf{b}}_{0_2}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} - 1\right)$,
$\beta_3 = 2k_1\left(\hat{\mathbf{b}}_{0_3}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} - 1\right)$,
$\beta_{11} = k_2\left(\hat{\mathbf{b}}_{0_1}^2\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} - 1\right)^2$,

$$\beta_{12} = k_2\left(\hat{\mathbf{b}}_{0_2}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} - 1\right)^2,$$
$$\beta_{13} = k_2\left(\hat{\mathbf{b}}_{0_3}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} - 1\right)^2$$

Expressing the components of the Cauchy stress the author obtained the following equations:

$$\begin{aligned}\sigma_1 = \frac{2\mu}{3}\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} + \frac{4k_1}{3}\hat{\mathbf{a}}_{0_1}^2\lambda_1^2 A_1 exp\left(k_2A_1^2\right)\\ +\frac{4k_1}{3}\hat{\mathbf{b}}_{0_1}^2\lambda_1^2 B_1 exp\left(k_2B_1^2\right)\end{aligned} \tag{16}$$

$$\begin{aligned}\sigma_2 = \frac{2\mu}{3}\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} + \frac{4k_1}{3}\hat{\mathbf{a}}_{0_2}^2\lambda_2^2 A_2 exp\left(k_2A_2^2\right)\\ +\frac{4k_1}{3}\hat{\mathbf{b}}_{0_2}^2\lambda_2^2 B_2 exp\left(k_2B_2^2\right)\end{aligned} \tag{17}$$

$$\begin{aligned}\sigma_3 = \frac{2\mu}{3}\lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} + \frac{4k_1}{3}\hat{\mathbf{a}}_{0_3}^2\lambda_3^2 A_3 exp\left(k_2A_1^2\right)\\ +\frac{4k_1}{3}\hat{\mathbf{b}}_{0_3}^2\lambda_3^2 B_3 exp\left(k_2B_3^2\right)\end{aligned} \tag{18}$$

where:

$$A_1 = \hat{\mathbf{a}}_{0_1}^2\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} - 1 = 25\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} - 1 \tag{19}$$

$$B_1 = \hat{\mathbf{b}}_{0_1}^2\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} - 1 = 9\lambda_1^{\frac{4}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{-\frac{2}{3}} - 1 \tag{20}$$

$$A_2 = \hat{\mathbf{a}}_{0_2}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} - 1 = 4\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} - 1 \tag{21}$$

$$B_2 = \hat{\mathbf{b}}_{0_2}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} - 1 = 4\lambda_1^{-\frac{2}{3}}\lambda_2^{\frac{4}{3}}\lambda_3^{-\frac{2}{3}} - 1 \tag{22}$$

$$A_2 = B_2, \quad \hat{\mathbf{a}}_{0_2}^2 = \hat{\mathbf{b}}_{0_2}^2 \tag{23}$$

$$A_3 = \hat{\mathbf{a}}_{0_3}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} - 1 = \lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} - 1 \tag{24}$$

$$B_3 = \hat{\mathbf{b}}_{0_3}^2\lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} - 1 = \lambda_1^{-\frac{2}{3}}\lambda_2^{-\frac{2}{3}}\lambda_3^{\frac{4}{3}} - 1 \tag{25}$$

$$A_3 = B_3, \quad \hat{\mathbf{a}}_{0_3}^2 = \hat{\mathbf{b}}_{0_3}^2 \tag{26}$$

The following values are assumed for μ, $\hat{\mathbf{a}}_0$, $\hat{\mathbf{b}}_0$, μ and $k_1 and$ k_2 are substituted in the above equations:

$$\sigma = \begin{bmatrix}18 & 0 & 0\\ 0 & 48 & 0\\ 0 & 0 & 18\end{bmatrix}, \hat{\mathbf{a}}_0 = \begin{bmatrix}5\\2\\1\end{bmatrix}, \hat{\mathbf{b}}_0 = \begin{bmatrix}3\\2\\1\end{bmatrix} \tag{27}$$

$$\mu = 0.8, \quad k_1 = 12.1, \quad k_2 = 14.2 \tag{28}$$

After performing logarithmization of equation (16-18) the obtained system with the stretches λ_1, λ_2, λ_3 is as follows:

$$-2ln\lambda_1 + ln\lambda_2 + ln\lambda_3 = 0.94856 \tag{29}$$

$$ln\lambda_1 - 2ln\lambda_2 + ln\lambda_3 = -0.863611 \tag{30}$$

$$ln\lambda_1 + ln\lambda_2 - 2ln\lambda_3 = -1.759491 \tag{31}$$

The equation (32) is derived by subtracting equation (26) and equation (27):

$$ln\lambda_2 = ln\lambda_3 - 0.298627 \tag{32}$$

The equation (33) is obtained by substituting equation (32) in equation (25):

$$ln\lambda_3 = ln\lambda_1 + 0.623594 \tag{33}$$

Substituting equation (32) and equation (29) in equation (25) gives equations for $ln\lambda_1$ and λ_1 :

$$ln\lambda_1 = -0.902686 \tag{34}$$

$$\lambda_1 = 0.405248 \tag{35}$$

The equations for $ln\lambda_3$ and λ_3 are obtained by substituting equation (34) in equation (33):

$$ln\lambda_3 = -0.279092 \tag{36}$$

$$\lambda_1 = 0.756337 \tag{37}$$

Substituting equation (36) in equation (32) gives equation (38) and equation (39) for $n\lambda_2$ and λ_2, correspondingly:

$$ln\lambda_2 = -0.577719 \tag{38}$$

$$\lambda_2 = 0.560972 \tag{39}$$

3. CONCLUSION

In this paper Holzapfel anisotropic model reporting mechanobiological behavior of deseased arterial tissues is considered. The system of the obtained equations for Cauchy stress is solved for definite constants. The obtained appropriate values of stretch ratios proves the adequacy of the applied model and the material parameters taken into account.

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