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RESEARCH ARTICLE

On Evolutionary Games with Periodic Payoffs

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ABSTRACT

Two cases of evolutionary stable strategy with periodic payoffs are studied. The first is a generalization of Uyttendaele et al. The second is prisoner's dilemma with periodic payoff. It is shown that reducing the defection payoff by a periodic term is insufficient to introduce cooperation. The study is generalized to 3-strategies games.

Keywords: Evolutionary games, Periodic payoffs, Prisoner’s dilemma, Co-operation, Strategies.

1. INTRODUCTION

Seasonality is an important phenomena in biology. Yet studying evolutionary games [1,2] with periodic payoffs still needs further work [3, 4, 5, 6, 7].

In 2.1 the work of [3] will be generalized to the case $\sigma << 1 and \alpha = 1$ which has not been studied. In section 2.2 the case of prisoner's dilemma game with periodic payoff will be introduced. The study is generalized to 3-strategies games in section 2.3

2. RESULTS

2.1. Generalized [3] case

[3] have studied evolutionary game with payoff matrix shown in (2.1)

$$A = \begin{bmatrix} 0 & 0 & 2+\sigma\cos(\rho t) \\ \propto & 0 & \alpha \\ 2-\sigma\cos(\rho t) & 0 & 0 \end{bmatrix} \qquad (2.1)$$

numerically where α, σ, ρ are positive constants. The replictor dynamics equations are

$$\frac{dp_i}{dt} = p_i\left[(A\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix})_i - p^t A p \right], i = 1,2,3$$

$$p_{1+}p_2 + p_3 = 1 \qquad (2.2)$$

We will study the case $\sigma << 1$ which was not considered in their paper. The equilibrium solution (p_i become constants for all i=1, 2, 3) of (2.2) implies (2.3):

$$\propto = 2p_3/(p_1+p_3), 2p_1 + p_2 = 1 \Rightarrow p_1 = p, p_3 = p, p_2 = 1-2p, \propto = 1 \qquad (2.3)$$

To study the time dependent case for

$$p_1 = p_3 = p_0 + \sigma x(t), p_2 = 1 - 2p_0 - 2\sigma x(t) \qquad (2.4)$$

Substituting (2.4) in (2.2) and linearizing in σ to find x (t) we get (2.5) and (2.6)

$$\frac{dx}{dt} = p_0^2\cos(\rho t), 0 < p_0 < \min\left\{\frac{1}{2}, \frac{\rho}{\sigma}\right\} \qquad (2.5)$$

Hence

$$p_1 \approx p_3 \approx p_0 + \left(\frac{\sigma p_0^2}{\rho}\right)\sin(\rho t) + O(\sigma^2)$$

$$p_2 \approx 1 - 2p_0 - 2\left(\frac{\sigma p_0^2}{\rho}\right)\sin(\rho t) + O(\sigma^2) \qquad (2.6)$$

In this case all strategies co-exist.

2.2. Prisoner’s dilemma with periodic payoff

Prisoner's dilemma game is a classic model for cooperation between selfish individuals [1]. Its payoff matrix is given by equation (2.7)

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$$A = \begin{bmatrix} R & S \\ T & U \end{bmatrix} T > R > U > S, 2R > T + S \quad (2.7)$$

The evolutionary stable strategy is defect (p=0). The reason for this negative result is the condition U>S. This negative result has been corrected by the following mechanisms: Relatedness, memory and inhomogeneity [8]. From the symmetry of the replicator dynamics [1] one can set S=0 without loss of generality.

Here we ask the following question: If a periodic term is added to reduce U will this introduce a kind of cooperation? Consider PD game with the following payoff matrix

$$A = \begin{bmatrix} R & 0 \\ T & U_0 - U_1\cos(t) \end{bmatrix}, U_1 > U_0 \quad (2.8)$$

The replicator dynamics equation is

$$dp/dt = p(1-p)[p(R - T + U_0 - U_1\cos(t)) - U_0 + U_1\cos(t)] \quad (2.9)$$

Studying (2.9) near defection p=0 one gets (2.10)

$$\frac{dp}{dt} \approx p[-U_0 + U_1\cos(t)] \Rightarrow p = p_0\exp(-U_0 t + U_1\sin(t)), \; 0 < p_0 < 1 \quad (2.10)$$

To study the system more formally consider a 2 strategies evolutionary game with payoff matrix Π. The corresponding equation is:

$$dp/dt = p(1-p)[p(\Pi\_11 + \Pi\_22 - \Pi\_12 - \Pi\_21) + (\Pi\_12 - \Pi\_22)] \quad (2.11)$$

In the case of time dependent payoff there are still two equilibrium solutions p=0, p=1. Solving for the stability near p=0 one gets (2.12)

$$dp/dt \approx p[\Pi_{12} - \Pi_{22}] \quad (2.12)$$

For p close to 1 set p=1-x, one gets

$$dx/dt \approx x[\Pi_{21} - \Pi_{11}) \quad (2.13)$$

For the case of prisoner's dilemma equation (2.13) becomes (2.14)

$$dx/dt \approx x[T - R] \quad (2.14)$$

Therefore, for the matrix (2.8) cooperation (p=1) is unstable.

Therefore, for the payoff matrix (2.8), reducing the defection payoff by a periodic term is not enough to introduce cooperation.

Now we derive the sufficient condition for the stability of the periodic solution of (2.11). Here we use a method similar to the one used in [9] where we approximate

$$\frac{dp(t)}{dt} \approx p(t + T1) - p(t) \forall t > 0 \quad (2.15)$$

And the system (2.11) is replaced by the finite difference equation in (2.16)

$$p(t + T1) \approx p(t) + p(t)\big(1 - p(t)\big)[p(t)(\Pi_{11} + \Pi_{22} - \Pi_{12} - \Pi_{21}) + (\Pi_{12} - \Pi_{22})] \quad (2.16)$$

Hence we get,

Proposition (1): The sufficient condition for the periodic solution of (2.11) to be stable is given in equation (2.17)

$$p(t)\big(1 - p(t)\big)[p(t)\Pi_{11} + \Pi_{22} - \Pi_{12} - \Pi_{21}) + (\Pi_{12} - \Pi_{22})] < 0 \forall T1 > t \geq 0 \quad (2.17)$$

This method has been successfully used [9] to derive Lyapunov conditions and sufficient conditions for the stability of the periodic solutions of Hill's and Lame' equation

2.3. Games with three strategies

In this case the replicator equation takes the form

$$dp_1/dt = p_1[\Pi_{11}p_1 + \Pi_{12}p_2 + \Pi_{13}(1 - p_1 - p_2) - \Pi_{11}(p_1)^2 - (\Pi_{12} + \Pi_{21})p_1p_2 - (\Pi_{13} + \Pi_{31})p_1(1 - p_1 - p_2) - \Pi_{22}(p_2)^2 - (\Pi_{23} + \Pi_{32})p_2(1 - p_1 - p_2) - \Pi_{33}(1 - p_1 - p_2)^2]$$

$$dp_2/dt = p_2[\Pi_{21}p_1 + \Pi_{22}p_2 + \Pi_{23}(1 - p_1 - p_2) - \Pi_{11}(p_1)^2 - (\Pi_{12} + \Pi_{21})p_1p_2 - (\Pi_{13} + \Pi_{31})p_1(1 - p_1 - p_2) - \Pi_{22}(p_2)^2 - (\Pi_{23} + \Pi_{32})p_2(1 - p_1 - p_2) - \Pi_{33}(1 - p_1 - p_2)^2]$$

where the elements of the payoff matrix are functions of time.

For the case of Hawk-Dove Retaliate [10] game the payoff matrix is given by (2.18)

$$\Pi = \begin{bmatrix} (v-c)/2 & v & (v-c)/2 \\ 0 & v/2 & v/2 \\ (v-c)/2 & v/2 & v/2 \end{bmatrix}, \quad c > v > 0 \qquad (2.18)$$

For the time independent case it is known that retaliate is the stable strategy. Studying the stability near the retaliate strategy $(p_1 = p_2 = 0)$ one gets $dp_1/dt \approx -c(t)p_1/2, dp_2/dt \approx 0$ where the cost c is considered to be function of time i.e. c=c(t). Thus we obtain

Proposition (2): For Hawk-Dove retaliate game (2.15) retaliate is the stable strategy if c(t)>0 for all T1>t>0.

3. DISCUSSION AND CONCLUSION

From the above results it is clear that periodic payoffs is more realistic than static ones. They can, in some cases, significantly change the outcome of the game.

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